

Warped products and Spaces of Constant Curvature

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Abstract

We will obtain the warped product decompositions of spaces of constant curvature (with arbitrary signature) in their natural models as subsets of pseudo-Euclidean space. This generalizes the corresponding result by Nolker in [Nol96] to arbitrary signatures, and has a similar level of detail. Although our derivation is complete in some sense, none is proven. Motivated by applications, we will give more information for the spaces with Euclidean and Lorentzian signatures. This is an expository article which is intended to be used as a reference. So we also give a review of the theory of circles and spheres in pseudo-Riemannian manifolds.

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List of Notations

- \mathbb{E}_ν^n pseudo-Euclidean space, an n -dimensional vector space equipped with a metric with signature ν [iii](#), [2](#), [20](#)
- $\mathbb{E}_\nu^n(\kappa)$ A hyperquadric of pseudo-Euclidean space. More precisely the central hyperquadric of \mathbb{E}_ν^n with curvature κ . [3](#)
- sgn Given a real number a , $\operatorname{sgn} a$ is the sign of a if $a \neq 0$ and 0 if $a = 0$. [3](#)

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1 Introduction

Warped products are ubiquitous in applications of pseudo-Riemannian geometry. Most of the separable coordinate systems in spaces of constant curvature are built up using them [Kal86], and some exact solutions in general relativity are composed of them [DU05; Zeg11]. They can intuitively be thought of as a partial generalization of the spherical coordinate system to arbitrary pseudo-Riemannian manifolds. Indeed, it can be shown that all the spherical coordinate systems (on any space of constant curvature) can be constructed iteratively using warped products, and that they share several properties with these coordinate systems. Similarly the well known Schwarzschild metric in relativity can be constructed using warped products.

Various geometrical objects take canonical forms in warped products. For example, one can calculate general formulas for the Levi-Civita connection and the Riemann curvature tensor in a warped product [MRS99]. These product manifolds can be used to construct geometrical objects with special properties. For example, it was shown in [RM14], that one can use the warped product decompositions of a given space to construct Killing tensors and hence coordinates which separate the Hamilton-Jacobi equation. Thus it is only natural that we determine the warped products which are isometric to spaces of constant curvature.

We now describe more precisely the problem we solve, after introducing some definitions. A *warped product* is a product manifold $M = \prod_{i=0}^k M_i$ of pseudo-Riemannian manifolds (M_i, g_i) where $\dim M_i > 0$ for $i > 0$ equipped with the metric

$$g = \pi_0^* g_0 + \sum_{i=1}^k \rho_i^2 \pi_i^* g_i$$

where $\rho_i : M_0 \rightarrow \mathbb{R}^+$ are functions and $\pi_i : M \rightarrow M_i$ are the canonical projection maps [MRS99]. The warped product is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$. We say a warped product is a *warped product decomposition* of a pseudo-Riemannian manifold M if it is isometric to some non-empty open subset of M . In this article we will present an interesting class of warped product decompositions of spaces of constant curvature (with arbitrary signature).

Our solution follows that by Nolker in [Nol96], which is for the special case of Riemannian spaces of constant curvature. We make use of the observation that for a warped product $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$, M_0 is a *geodesic submanifold* of M and for each $i > 0$ the manifold M_i is a *spherical submanifold* of M ¹ [MRS99]. Thus after characterizing all geodesic and spherical submanifolds of spaces of constant curvature, we generalize a formula given in [Nol96] to obtain an interesting class of warped product decompositions.

Our primary motivation for this work comes from [RM14], where it was shown that warped products can be used to construct coordinates which separate the Hamilton-Jacobi equation. Based on this application, it will become clear (in a following article)

¹Often a geodesic submanifold is called totally geodesic and a spherical submanifold is called an extrinsic sphere.

that our work is “complete”. We are mainly interested in exposing these results for reference purposes. Since as of now, it is difficult to find any articles/books which can be used as a reference for our purposes. For a similar reason, we will present a review of the theory of circles and spheres in pseudo-Riemannian manifolds as well.

This article is mostly self-contained, so it can be used as a reference. However, we use some results from the theory of pseudo-Riemannian submanifolds in [Che11], which are only necessary to understand certain proofs. We also assume the reader is familiar with [O’N83], especially with the basic properties of pseudo-Euclidean vector spaces and (pseudo-)Riemannian submanifold theory. Familiarity with the article [Nol96] is useful but not necessary.

The article is organized as follows. After defining basic notations in Section 2, we summarize our results in Section 3. This summary should be sufficient for applications. The subsequent sections provide proofs and more details. In Section 4 we give a brief review of the theory of pseudo-Riemannian submanifolds/foliations. In Section 5 we apply these concepts by reviewing the theory of circles and spheres in pseudo-Riemannian manifolds. This section is optional but it gives a geometric interpretation of warped products and is included because there are relatively few reviews of this topic. In Sections 6 to 9 we review preliminary theory on the spherical submanifolds and warped products in spaces of constant curvature and warped products in general. We give the warped product decompositions of pseudo-Euclidean space in Section 10 and of spherical submanifolds of pseudo-Euclidean space in Section 12. Section 11 is another optional section which gives the isometry groups of spherical submanifolds of pseudo-Euclidean space, referring to [O’N83] in the appropriate cases.

2 Notations and Conventions

All differentiable structures are assumed to be smooth (class C^∞). Let M be a pseudo-Riemannian manifold of dimension n equipped with covariant metric g . Unless specified otherwise, it is assumed that $n \geq 2$. The contravariant metric is usually denoted by G and $\langle \cdot, \cdot \rangle$ plays the role of the covariant and contravariant metric depending on the arguments. We denote by $\mathcal{F}(M)$ the set of functions from M to \mathbb{R} and $\mathfrak{X}(M)$ denotes the set of vector fields over M . If $x \in \mathfrak{X}(M)$ then we denote $x^2 := \langle x, x \rangle$ and $\|x\| := \sqrt{|\langle x, x \rangle|}$.

Throughout this article we will be working in pseudo-Euclidean space, which is defined as follows. An n -dimensional vector space V equipped with metric g of signature² ν is denoted by \mathbb{E}_ν^n and called *pseudo-Euclidean space*. We obtain Euclidean space \mathbb{E}^n in the special case where $\nu = 0$. Also Minkowski space M^n is obtained by taking $\nu = 1$. A subspace $U \subseteq \mathbb{E}_\nu^n$ is called *non-degenerate* if the induced metric is non-degenerate and in this case we denote by $\text{ind } U$ the signature of the induced metric. Also note that since \mathbb{E}_ν^n is a vector space, for any $p \in \mathbb{E}_\nu^n$ we identify vectors in $T_p \mathbb{E}_\nu^n$ with points in \mathbb{E}_ν^n .

²The signature is equal to the number of negative diagonal entries in a basis which diagonalizes g .

Given an open subset $U \subseteq \mathbb{E}_\nu^n$ and $\kappa \in \mathbb{R} \setminus \{0\}$, we denote by $U(\kappa)$ the *central hyperquadric* of \mathbb{E}_ν^n contained in U , which is defined by:

$$U(\kappa) = \{p \in U \mid \langle p, p \rangle = \kappa^{-1}\}$$

Usually $U = \mathbb{E}_\nu^n$ and this is denoted $\mathbb{E}_\nu^n(\kappa)$. The notation $U(\kappa)^\circ$ represents a maximal connected component of $U(\kappa)$. It is well known that $\mathbb{E}_\nu^n(\kappa)$ is a pseudo-Riemannian manifold of dimension $n - 1$ with signature $\nu + \frac{(\text{sgn } \kappa - 1)}{2}$ and constant curvature³ κ [O’N83]. Since $\mathbb{E}_\nu^n(\kappa) \subset \mathbb{E}_\nu^n$, for any $p \in \mathbb{E}_\nu^n(\kappa)$ we identify vectors in $T_p \mathbb{E}_\nu^n(\kappa)$ with points in \mathbb{E}_ν^n . Occasionally we use the following conventions: If $\kappa = 0$, we set $\mathbb{E}_\nu^n(0) := \mathbb{E}_\nu^n$, if $\kappa = \infty$ we set $\mathbb{E}_\nu^n(\infty)$ to be the light cone, i.e. the set of non-zero null vectors. We also use the following notations: If $\kappa > 0$ then $S_\nu^n(\kappa) := \mathbb{E}_\nu^{n+1}(\kappa)^\circ$, if $\kappa < 0$ then $H_\nu^n(\kappa) := \mathbb{E}_{\nu+1}^{n+1}(\kappa)^\circ$.

We define the parabolic embedding of \mathbb{E}_ν^n in $\mathbb{E}_{\nu+1}^{n+2}$ with mean curvature vector $-a \in \mathbb{E}_\nu^n(\infty)$ by [Toj07]

$$\mathbb{P}_\nu^n := \{p \in \mathbb{E}_{\nu+1}^{n+2}(\infty) : \langle p, a \rangle = 1\}$$

An explicit isometry with \mathbb{E}_ν^n is obtained by choosing $b \in \mathbb{P}_\nu^n$, i.e. b is lightlike and $\langle a, b \rangle = 1$. We let $V := \text{span}\{a, b\}^\perp$, note that $V \cong \mathbb{E}_\nu^n$, then for $x \in V$:

$$\psi(x) = b + x - \frac{1}{2}x^2 a \in \mathbb{P}_\nu^n \quad (2.3)$$

More details on the properties of \mathbb{P}_ν^n will be given later on (see Proposition 7.2). Finally, we define the *dilatational vector field* in \mathbb{E}_ν^n , r , to be the vector field satisfying for any $p \in \mathbb{E}_\nu^n$, $r_p = p \in T_p \mathbb{E}_\nu^n$.

3 Warped products in Spaces of Constant Curvature

In this section we will briefly describe the warped product decompositions of spaces of constant curvature, in a way which is useful for applications. The proofs of many of the assertions will come in the following sections. We will use the notation $\mathbb{E}_\nu^n(\kappa)$ (where κ can be zero) to represent the general space of constant curvature. First we will need to know the spherical submanifolds of these spaces.

Theorem 3.1 (Spherical submanifolds of $\mathbb{E}_\nu^n(\kappa)$)

Let $\bar{p} \in \mathbb{E}_\nu^n(\kappa)$ be arbitrary, $V \subset T_{\bar{p}} \mathbb{E}_\nu^n(\kappa)$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \text{ind } V$ and $z \in V^\perp \cap T_{\bar{p}} \mathbb{E}_\nu^n(\kappa)$. Let $a := \kappa \bar{p} - z$, $\tilde{\kappa} := a^2$ and $W := \mathbb{R}a \oplus V$. There is exactly one m -dimensional connected and geodesically complete spherical submanifold \tilde{N} with $\bar{p} \in \tilde{N}$, $T_{\bar{p}} \tilde{N} = V$ and having mean curvature vector at \bar{p} , z . \tilde{N} is an open submanifold of N ; N is referred to as the spherical submanifold determined by (\bar{p}, V, a) , it is geodesic iff $z = 0$ and is given as follows (where \simeq means isometric to):

³This will be proven later.

(a) $a = 0$, in this case $N \simeq \mathbb{E}_\mu^m$

$$N = \bar{p} + V$$

(b) a is timelike, then $\mu \leq \nu - 1$ and $N \simeq H_\mu^m(\tilde{\kappa})$

(c) a is spacelike, then $N \simeq S_\mu^m(\tilde{\kappa})$

For cases (b) and (c), let $c = \bar{p} - \frac{a}{\tilde{\kappa}}$ be the center of N , then N is given as follows:

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(d) a is lightlike, then $\mu \leq \nu - 1$ and $N \simeq \mathbb{E}_\mu^m$

$$N = \bar{p} + \{p - \frac{1}{2}p^2a \mid p \in V\}$$

□

Remark 3.2

If a is lightlike, then N is isometric to \mathbb{P}_μ^m with mean curvature vector $-a$. Furthermore, let $b \in V^\perp$ be a lightlike vector satisfying $\langle a, b \rangle = 1$. Then the orthogonal projector onto V , P , induces an isometry of $N - \bar{p} + b$ onto V . □

Remark 3.3

One can find more details on when N is connected in the remarks following Theorems 10.1 and 12.2. □

PROOF See Theorems 10.1 and 12.2. ■

With the knowledge of these spherical submanifolds, we can now specify how to construct warped products in $\mathbb{E}_\nu^n(\kappa)$. This construction depends on the following data: A point $\bar{p} \in \mathbb{E}_\nu^n(\kappa)$, a decomposition $T_{\bar{p}}\mathbb{E}_\nu^n(\kappa) = \bigoplus_{i=0}^k V_i$ into non-trivial (hence non-degenerate) subspaces with $k \geq 1$, and vectors $z_1, \dots, z_k \in V_0$ such that the vectors $a_i := \kappa\bar{p} - z_i$ are pair-wise orthogonal and independent. We call the data $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$, *initial data* for a (proper) warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$. If $\kappa = 0$, one can more generally let some of the a_i be zero, this results in Cartesian products as done in [Nol96]. Since we assume the a_i are non-zero, we sometimes use the additional qualifier “proper”.

With this initial data, for $i > 0$ let N_i be the sphere in $\mathbb{E}_\nu^n(\kappa)$ determined by (\bar{p}, V_i, a_i) and $\rho_i(p_0) = 1 + \langle a_i, p_0 - \bar{p} \rangle$. Let N_0 be the subset of the sphere in $\mathbb{E}_\nu^n(\kappa)$

determined by $(\bar{p}, V_0, \kappa \bar{p})$ where each $\rho_i > 0$. Then the data $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$, induces a warped product decomposition (of $\mathbb{E}_\nu^n(\kappa)$) given as follows:

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k & \rightarrow \mathbb{E}_\nu^n(\kappa) \\ (p_0, \dots, p_k) & \mapsto p_0 + \sum_{i=1}^k \rho_i(p_i)(p_i - \bar{p}) \end{cases} \quad (3.4)$$

We note that ψ has the property that $\psi(\bar{p}, \dots, p_i, \bar{p}, \dots, \bar{p}) = p_i$. Often the point \bar{p} doesn't enter calculations, hence we will usually omit it. We note that the above formula generalizes one given in [Nol96].

For actual calculations, it will be more convenient to work with canonical forms. The following definition will be particularly useful.

Definition 3.4 (Canonical form for Warped products of \mathbb{E}_ν^n)

We say that a proper warped product decomposition of \mathbb{E}_ν^n determined by $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ is in canonical form if: $\bar{p} \in V_0$ and $\langle \bar{p}, a_i \rangle = 1$. \square

Any proper warped product decomposition ψ of \mathbb{E}_ν^n can be brought into canonical form, see the discussion preceding Corollary 10.10 for details.

We will now give more information on standard warped product decompositions of \mathbb{E}_ν^n in canonical form. Suppose the initial data $(\bar{p}; V_0 \oplus V_1; a)$ is in canonical form, and let ψ be the associated warped product decomposition given by Eq. (3.4). Denote $\kappa := a^2$ and $\epsilon := \text{sgn } \kappa$. We have two types of warped products:

non-null warped decomposition If $\kappa \neq 0$, let $W_0 := V_0 \cap a^\perp$ and $W_1 := W_0^\perp$.

null warped decomposition If $\kappa = 0$, then a is lightlike, so fix another lightlike vector $b \in V_0$ such that $\langle a, b \rangle = 1$, let $W_0 := V_0 \cap \text{span}\{a, b\}^\perp$ and $W_1 := V_1$.

For $i = 0, 1$, let $P_i : \mathbb{E}_\nu^n \rightarrow W_i$ be the orthogonal projection. Then the following holds:

Theorem 3.5 (Standard Warped Products in \mathbb{E}_ν^n [Nol96])

Let ψ be the warped product decomposition of \mathbb{E}_ν^n determined by the initial data $(\bar{p}; V_0 \oplus V_1; a)$ given above. Then N_0 has the following form:

$$N_0 = \{p \in V_0 \mid \langle a, p \rangle > 0\}$$

and

$$\rho : \begin{cases} N_0 & \rightarrow \mathbb{R}_+ \\ p_0 & \mapsto \langle a, p_0 \rangle \end{cases}$$

The map ψ is an isometry onto the following set:

$$\text{Im}(\psi) = \begin{cases} \{p \in \mathbb{E}_\nu^n \mid \text{sgn}(P_1 p)^2 = \epsilon\} & \text{non-null case} \\ \{p \in \mathbb{E}_\nu^n \mid \langle a, p \rangle > 0\} & \text{null case} \end{cases} \quad \square$$

Furthermore, the following equation holds:

$$\psi(p_0, p_1)^2 = p_0^2 \quad (3.8)$$

PROOF See Corollary 10.10. ■

In fact, for $(p_0, p_1) \in N_0 \times N_1$, ψ has one of the following forms, first if ψ is non-null:

$$\psi(p_0, p_1) = P_0 p_0 + \langle a, p_0 \rangle (p_1 - c)$$

where $c = \bar{p} - \frac{a}{a^2}$, and if ψ is null:

$$\psi(p_0, p_1) = P_0 p_0 + (\langle b, p_0 \rangle - \frac{1}{2} \langle a, p_0 \rangle (P_1 p_1)^2) a + \langle a, p_0 \rangle b + \langle a, p_0 \rangle P_1 p_1$$

The above forms are obtained from the equation for ψ from the above theorem by expanding p_0 in an appropriate basis. We note that the warped products with multiple spherical factors can be obtained using the standard ones described above. Indeed, suppose $\phi_1 : N'_0 \times_{\rho_1} N_1 \rightarrow \mathbb{E}_\nu^n$ is the warped product decomposition determined by $(\bar{p}; V_0 \oplus V_1; a_1)$ as above. Since V_0 is pseudo-Euclidean, consider a warped product decomposition, $\phi_2 : \tilde{N}_0 \times_{\rho_2} N_2 \rightarrow V_0$, determined by $(\bar{p}; \tilde{V}_0 \oplus \tilde{V}_1; a_2)$ with $V_0 \cap W_0^\perp \subset \tilde{W}_0$ (hence $a_1 \in \tilde{W}_0$). Note that \tilde{W}_0 is the subspace W_0 from the above construction for ϕ_2 . Let $N_0 := N'_0 \cap \tilde{N}_0$, then one can check that the map ψ defined by:

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times_{\rho_2} N_2 & \rightarrow \mathbb{E}_\nu^n \\ (p_0, p_1, p_2) & \mapsto \phi_1(\phi_2(p_0, p_2), p_1) \end{cases}$$

is a warped product decomposition of \mathbb{E}_ν^n satisfying Eq. (3.4). We illustrate this construction with an example.

Example 3.6 (Constructing multiply warped products)

Suppose ϕ_1 and ϕ_2 are given as follows:

$$\begin{aligned} \phi_1(p'_0, p_1) &= P'_0 p'_0 + \langle a_1, p'_0 \rangle (p_1 - c_1) \\ \phi_2(\tilde{p}_0, p_2) &= \tilde{P}_0 \tilde{p}_0 + \langle a_2, \tilde{p}_0 \rangle (p_2 - c_2) \end{aligned}$$

Now observe that $\rho_1(\phi_2(\tilde{p}_0, p_2)) = \rho_1(\tilde{p}_0)$, which follows from the above equation for ϕ_2 and the fact that $a_1 \in \tilde{W}_0$. Then,

$$\begin{aligned}
\psi(p_0, p_1, p_2) &= \phi_1(\phi_2(p_0, p_2), p_1) \\
&= P'_0 \phi_2(p_0, p_2) + \langle a_1, \phi_2(p_0, p_2) \rangle (p_1 - c_1) \\
&= P'_0 \tilde{P}_0 p_0 + \langle a_2, p_0 \rangle (p_2 - c_2) + \langle a_1, p_0 \rangle (p_1 - c_1)
\end{aligned}$$

where $P'_0 \tilde{P}_0$ is the orthogonal projector onto $\tilde{W}_0 \cap W_0 = \tilde{V}_0 \cap \text{span}\{a_1, a_2\}^\perp$. A similar calculation shows that ψ satisfies Eq. (3.4), since ϕ_1 and ϕ_2 each satisfy it. \square

This procedure can be repeated as many times as necessary to obtain the more general warped products given by Eq. (3.4). Hence the properties of the more general warped product decompositions of \mathbb{E}_ν^n can be deduced from Theorem 3.5.

The following proposition shows that any proper warped product decomposition of \mathbb{E}_ν^n in canonical form restricts to a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ where $\kappa \neq 0$. Its proof is straightforward consequence of Eq. (3.8).

Theorem 3.7 (Restricting Warped products to $\mathbb{E}_\nu^n(\kappa)$)

Let ψ be a proper warped product decomposition of \mathbb{E}_ν^n associated with $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ in canonical form. Suppose $\kappa^{-1} := \bar{p}^2 \neq 0$ and let $N' := N_0(\kappa) \times_{\rho_1} N_1 \times \dots \times_{\rho_k} N_k$. Then $\phi : N' \rightarrow \mathbb{E}_\nu^n(\kappa)$ defined by $\phi := \psi|_{N'}$ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ passing through \bar{p} . \square

PROOF See Theorem 12.5. \blacksquare

Hence the details of warped product decompositions of $\mathbb{E}_\nu^n(\kappa)$ can be deduced from Theorem 3.5. More information on these decompositions can be found in the following sections. In particular, see Theorems 10.5 and 12.8. Some examples can be found in [Nol96] and also in a future article where we apply these results to construct coordinates which separate the Hamilton-Jacobi equation.

4 pseudo-Riemannian Submanifolds and Foliations

In this section we will summarize the theory of pseudo-Riemannian submanifolds and foliations that will be useful to us. We can conveniently treat this as a special case of the theory of pseudo-Riemannian distributions, so we will present this first. For more details on pseudo-Riemannian submanifolds see (for example) [O'N83; Lee97]. Similarly for pseudo-Riemannian foliations see [Rov98; Ton88].

4.1 Brief outline of The Theory of Pseudo-Riemannian Distributions

The following brief exposition of the theory of pseudo-Riemannian distributions is a combination of that given in [MRS99] and [CFS06]. Suppose E is an m -dimensional non-degenerate distribution defined on a pseudo-Riemannian manifold \bar{M} . Then we

use the orthogonal splitting $T\bar{M} = E \oplus E^\perp$, $V = V^E + V^{E^\perp}$, to define a tensor $s^E : T\bar{M} \times E \rightarrow E^\perp$ and a linear connection ∇^E for E by:

$$\nabla_X Y = \nabla_X^E Y + s^E(X, Y)$$

for all $X \in \mathfrak{X}(\bar{M})$ and $Y \in \Gamma(E)$. s^E is called the *generalized second fundamental form* of E and the above equation is referred to as the *Gauss equation*. One can also check that ∇^E is metric compatible, i.e. $X \langle Y, Z \rangle = \langle \nabla_X^E Y, Z \rangle + \langle Y, \nabla_X^E Z \rangle$ for all $X \in \mathfrak{X}(\bar{M})$ and $Y, Z \in \Gamma(E)$.

For the remainder of the discussion we set $s^E := s^E|_{(E \times E)}$. For $X, Y \in \Gamma(E)$, we can further decompose $s^E(X, Y)$ into its anti-symmetric and symmetric parts

$$\begin{aligned} s^E(X, Y) &= (\nabla_X Y)^{E^\perp} = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^{E^\perp} + \frac{1}{2}(\nabla_X Y - \nabla_Y X)^{E^\perp} \\ &= h^E(X, Y) + A^E(X, Y) \\ A^E(X, Y) &:= \frac{1}{2}(\nabla_X Y - \nabla_Y X)^{E^\perp} \\ h^E(X, Y) &:= \frac{1}{2}(\nabla_X Y + \nabla_Y X)^{E^\perp} \end{aligned}$$

Since ∇ is torsion-free, $A^E(X, Y) = \frac{1}{2}([X, Y])^{E^\perp}$, hence E is integrable iff $A^E \equiv 0$. h^E is called the *second fundamental form* of E . The second fundamental form can be decomposed in terms of its trace to get a further classification of E as follows:

$$\begin{aligned} h^E(X, Y) &= \langle X, Y \rangle H_E + h_T^E(X, Y) \\ H_E &= \frac{1}{m} \text{tr}(h^E) \end{aligned}$$

where h_T^E is trace-less. H_E is called the *mean curvature normal* of E . E is called *minimal*, *umbilical* or *geodesic*⁴ if $s^E(X, Y) = h_T^E(X, Y)$, $s^E(X, Y) = \langle X, Y \rangle H_E$ or $s^E(X, Y) = 0$ respectively for all $X, Y \in \Gamma(E)$. We add the qualification “almost” to the three definitions above by replacing s^E with h^E ; this just drops the requirement that $A^E \equiv 0$. For example E is *almost umbilical* iff $h_T^E = 0$. We remark that when E is one dimensional $h_T^E = 0$ trivially, hence all one dimensional non-degenerate foliations and all one dimensional pseudo-Riemannian submanifolds are trivially umbilical. If E is umbilical and $\nabla_X^{E^\perp} H_E = 0$ for all $X \in \Gamma(E)$ then E is called *spherical*. Finally if E is spherical and E^\perp is geodesic then E is called *Killing*.

We also note here that s^E and s^{E^\perp} are not independent of each other:

Proposition 4.1

For $X, Y \in \Gamma(E)$ and $Z \in \Gamma(E^\perp)$, the following holds:

$$\langle s^E(X, Y), Z \rangle = - \langle Y, s^{E^\perp}(X, Z) \rangle$$

□

⁴Note that some authors use the name auto-parallel instead [MRS99].

PROOF

$$\begin{aligned}
0 &= \nabla_X \langle Y, Z \rangle \\
&= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\
&= \langle s^E(X, Y), Z \rangle + \langle Y, s^{E^\perp}(X, Z) \rangle
\end{aligned}$$

■

4.2 Specialization to pseudo-Riemannian submanifolds

Suppose $\phi : M \rightarrow \bar{M}$ is a local embedding of (a pseudo-Riemannian submanifold) M^m inside \bar{M}^n . Then for any point $p \in M$, it is known that there exist local coordinates (x^i) on \bar{M} , such that the subset

$$\{(x^1, \dots, x^m, x^{m+1}, \dots, x^n) : x^{m+1} = c_{m+1}, \dots, x^n = c_n\}$$

for some $c_{m+1}, \dots, c_n \in \mathbb{R}$ can be identified with $\phi(U)$ where U is an open subset with $p \in U \subseteq M$. These coordinates induce a local foliation L in a neighborhood of p , with M being a leaf given by the above equation. We will refer to such a foliation as a (local) foliation of \bar{M} associated with M . Now suppose L is an arbitrary foliation of \bar{M} associated with M , and let E be the induced distribution. Locally we can assume L is a foliation by pseudo-Riemannian submanifolds of \bar{M} , hence E is non-degenerate and the discussion in the previous section applies to it. Since E is integrable, it follows that for any $X, Y \in \Gamma(E)$, that $[X, Y] \in \Gamma(E)$. Throughout this discussion, for any $X \in \Gamma(E)$, we let $\tilde{X} \in \mathfrak{X}(M)$ denote the unique vector field such that for any $p \in M$, we have $X_{\phi(p)} = \phi_* \tilde{X}_p$. Then for any $X, Y \in \Gamma(E)$ we see that

$$[X, Y]|_{\phi(p)} = \phi_*[\tilde{X}, \tilde{Y}]|_p$$

Thus $[X, Y]|_{\phi(p)}$ depends only on $[\tilde{X}, \tilde{Y}]|_p$ in M .

Now denote by ∇ (resp. $\bar{\nabla}$) the Levi-Civita connection on M (resp. \bar{M}). By the uniqueness properties of the Levi-Civita connection on M , it follows that for any $X, Y \in \Gamma(E)$ we have for any $p \in M$ that

$$(\bar{\nabla}_X^E Y)|_{\phi(p)} = \phi_*(\nabla_{\tilde{X}} \tilde{Y})|_p$$

Thus $(\bar{\nabla}_X^E Y)|_{\phi(p)}$ depends only on $(\nabla_{\tilde{X}} \tilde{Y})|_p$ in M . By also using the Gauss equation, we observe that for any $p \in M$, that $(\bar{\nabla}_X Y)|_{\phi(p)}$ depends only on \tilde{X} and \tilde{Y} .

In consequence of these observations, it follows that the theory presented for pseudo-Riemannian distributions induces a similar one for pseudo-Riemannian manifolds. We now connect this with the standard notations [Che11]; in effect this removes the appearance of the extraneous distribution, E .

In this case $s^E \equiv h^E$ and $h := (h^E)|_M$, then the Gauss equation becomes:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$. We denote the set of normal vector fields over M , i.e. the restriction of $\Gamma(E^\perp)$ to M by $\mathfrak{X}(M)^\perp$. The Gauss equation for E^\perp is usually called the *Weingarten equation* and is only defined for $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)^\perp$. This is because in this case, $\bar{\nabla}_X Y$ depends only on the values that X and Y take on M ⁵. Thus we can let $A_Y(X) := -s^{E^\perp}(X, Y)$, $\nabla_X^\perp Y := \bar{\nabla}_X^{E^\perp} Y$ and the Gauss equation (for E^\perp) becomes:

$$\bar{\nabla}_X Y = \nabla_X^\perp Y - A_Y(X)$$

for all $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)^\perp$. Note that the properties of $\bar{\nabla}^{E^\perp}$ imply that ∇^\perp is a connection⁶ on $\mathfrak{X}(M)^\perp$. In this notation, the relationship between s^E and s^{E^\perp} given in Proposition 4.1 becomes:

$$\langle h(X, Y), Z \rangle = \langle A_Z(X), Y \rangle \quad (4.8)$$

for all $X, Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M)^\perp$.

Finally, we note that the definitions of minimal, umbilical, or geodesic foliations induces corresponding definitions for submanifolds. For example, a submanifold is geodesic if its second fundamental form vanishes identically.

In conclusion, we should mention that even though we have given a concise presentation of the theory, it's not useful for practical calculations. For these, one will have to evaluate these quantities in terms of curves on M . See for example, Proposition 4.8 in [O'N83].

5 Circles and Spherical Submanifolds*

In this section we will briefly overview the theory of circles and spherical submanifolds of pseudo-Riemannian manifolds. Circles are covariantly defined using the Frenet formula, but the definition of a sphere requires more work [Nom73]. This material is not necessary to read the rest of the article, although it gives an application of the general theory presented in the previous section, a geometric interpretation of spherical submanifolds, and gives some background for the results on the intrinsic properties of warped products to come. We also present this theory here because it's not covered in standard references, in contrast with the corresponding theory for geodesic submanifolds (see [O'N83]).

A proper circle⁷ in a pseudo-Riemannian manifold is defined using the Frenet formula as a unit speed curve whose first curvature is constant and non-zero and remaining curvatures vanish. To be precise, let $\gamma(t)$ be a unit speed curve in M , i.e. $\dot{\gamma}^2 = \pm 1$. Let $X := \dot{\gamma}$. Let $\kappa(t) := \|\nabla_X X\|$ be the (first) curvature of γ . Assuming $\kappa \neq 0$, we

⁵This is because for any $p \in \bar{M}$, $(\bar{\nabla}_X Y)|_p$ depends only on the values of Y along any curve tangent to X_p . See Lemma 4.8 in [Lee97] and the following exercise, or Proposition 3.18 (3) in [O'N83].

⁶More precisely it satisfies the properties in definition 3.9 in [O'N83] and is metric compatible.

⁷Sometimes these are called geodesic circles [Ami03]. This name emphasizes the fact that we do not require the image of these curves to be a compact set, i.e. homeomorphic to S^1 .

define Y to be the unit vector field over γ derived from $\nabla_X X$. I.e. Y satisfies the following equation

$$\nabla_X X = \kappa Y$$

A proper circle is defined to be a curve which satisfies $\nabla_X Y = cX$ for some $c \in \mathbb{R} \setminus \{0\}$. We observe that

$$\begin{aligned} \langle \nabla_X Y, X \rangle &= -\langle Y, \nabla_X X \rangle \\ &= -\kappa \langle Y, Y \rangle \end{aligned}$$

The above equation implies that $c = -\varepsilon_0 \varepsilon_1 \kappa$ where $\varepsilon_0 := \operatorname{sgn} \langle X, X \rangle$ and $\varepsilon_1 := \operatorname{sgn} \langle Y, Y \rangle$. Thus a proper circle is defined by the equations

$$\begin{aligned} \nabla_X X &= \kappa Y \\ \nabla_X Y &= -\varepsilon_0 \varepsilon_1 \kappa X \end{aligned}$$

where $\kappa \neq 0$ is a constant. A proper circle satisfies the following third order ODE [ANY90]:

$$\nabla_X \nabla_X X = -\langle \nabla_X X, \nabla_X X \rangle \langle X, X \rangle X \quad (5.2)$$

Conversely we will see shortly that any unit speed curve satisfying the above equation with $\langle \nabla_X X, \nabla_X X \rangle \neq 0$ is a proper circle. We define a *circle* in a pseudo-Riemannian manifold to be a unit speed curve satisfying the above equation, hereafter called the *circle equation*. The following lemma shows that any pseudo-Riemannian manifold admits circles:

Lemma 5.1 (Existence and Uniqueness of Circles [NY74])

Consider the following initial conditions: $p \in M$, a unit vector $X_p \in T_p M$ and $Y_p \in X_p^\perp$. There exists a unique locally defined unit speed curve $\gamma(t)$ in M satisfying Eq. (5.2) and the initial conditions:

$$\begin{aligned} \gamma(0) &= p \\ \dot{\gamma}(0) &= X_p \\ (\nabla_X X)|_p &= Y_p \end{aligned}$$

where $X := \dot{\gamma}$ and $Y := \nabla_X X$. Furthermore, $\langle Y, Y \rangle$ is constant along any circle. \square

PROOF It follows by the existence and uniqueness theorem for ODEs that there exists a unique locally defined curve $\gamma(t)$ satisfying Eq. (5.2) with the above initial conditions. Then observe the following:

$$\begin{aligned}
\nabla_X \langle X, X \rangle &= 2 \langle X, \nabla_X X \rangle = \langle X, Y \rangle \\
\nabla_X \langle X, Y \rangle &= \langle Y, Y \rangle + \langle X, \nabla_X Y \rangle \\
&\stackrel{(5.2)}{=} \langle Y, Y \rangle - \langle X, X \rangle^2 \langle Y, Y \rangle \\
&= \langle Y, Y \rangle (\langle X, X \rangle^2 - 1)
\end{aligned}$$

The above two equations define a system of ODEs for $\langle X, X \rangle$ and $\langle X, Y \rangle$, with initial values $\langle X, X \rangle|_p = \varepsilon = \pm 1$ and $\langle X, Y \rangle|_p = 0$. Thus by the uniqueness of the solutions, it follows that $\langle X, X \rangle = \varepsilon$ and $\langle X, Y \rangle = 0$ wherever γ is defined. Hence γ is a unit speed curve.

Finally observe that

$$\begin{aligned}
\nabla_X \langle Y, Y \rangle &= 2 \langle \nabla_X Y, Y \rangle \\
&\stackrel{(5.2)}{=} -2 \langle \nabla_X X, \nabla_X X \rangle \langle X, X \rangle \langle X, Y \rangle \\
&= 0
\end{aligned}$$

Hence $\langle Y, Y \rangle$ is constant. ■

Note that $k := \|Y\|$ in the above lemma is usually called the *curvature* of the circle. In Riemannian manifolds, circles are completely classified by their curvature, although this is not true for pseudo-Riemannian manifolds. Using the above lemma we can classify circles in a pseudo-Riemannian manifold as follows. Let $\gamma(t)$ be a circle in M and suppose γ satisfies the initial conditions of the above lemma. Then γ can be classified as follows depending on Y_p :

Geodesic: If $Y_p = 0$.

Proper Circle: If $\langle Y, Y \rangle|_p \neq 0$.

Null Circle: If $\langle Y, Y \rangle|_p = 0$ but $Y_p \neq 0$, i.e. Y_p is lightlike, hence Eq. (5.2) reduces to $\nabla_X \nabla_X X = 0$.

Note that this classification is well defined globally since $\langle Y, Y \rangle$ is a constant of a circle and the uniqueness theorem for ODEs forces any circle with $Y_p = 0$ to be a geodesic.

Example 5.2 (Geodesics in Spherical Submanifolds [Kas10])

Let M be a spherical submanifold of \bar{M} . Suppose $\gamma(t)$ is a unit speed geodesic on M . We will show that γ is a circle in \bar{M} . By the Gauss equation, we have the following:

$$\bar{\nabla}_X X = \langle X, X \rangle H$$

Then by the Weingarten equation and using the fact that $\bar{\nabla}^\perp H = 0$ where $\bar{\nabla}^\perp$ is the induced normal connection over M , we have the following:

$$\begin{aligned}\bar{\nabla}_X \bar{\nabla}_X X &= \langle X, X \rangle \bar{\nabla}_X H \\ &= -\langle X, X \rangle A_H(X) \\ &= -\langle X, X \rangle \langle H, H \rangle X \\ &= -\langle X, X \rangle \langle \bar{\nabla}_X X, \bar{\nabla}_X X \rangle X\end{aligned}$$

since for any $Z \in \mathfrak{X}(M)^\perp$, $\langle A_H(X), Z \rangle \stackrel{(4.8)}{=} \langle h(X, Z), H \rangle = \langle X, Z \rangle \langle H, H \rangle$. \square

We note here that the above example in combination with Lemma 5.1 shows that the mean curvature vector field of a spherical submanifold is locally determined by its value at a single point. In Example 10.4 we will describe the circles in pseudo-Euclidean space after we have described the spherical submanifolds of the space.

We will now present some additional results that show how circles can be used to characterize spherical submanifolds. These results were first obtained for the Riemannian case by Nomizu and Yano in [NY74]. They were generalized to the Lorentzian case by Ikawa in [Ika85] and to the pseudo-Riemannian case by Abe, Nakanishi, and Yamaguchi in [ANY90].

For the following theorems we denote a pseudo-Riemannian manifold M with signature α by M_α . The following theorem characterizes spherical submanifolds in terms of circles, it is analogous to the corresponding theorem for geodesics and geodesic submanifolds (see [O'N83, section 4.4]).

Theorem 5.3 (Circles and Spheres [ANY90])

Let M_α be an n dimensional pseudo-Riemannian submanifold of \bar{M}_β . For any $\varepsilon_0 \in \{-1, 1\}$ and $\varepsilon_1 \in \{-1, 0, 1\}$ satisfying $2 - 2\alpha \leq \varepsilon_0 + \varepsilon_1 \leq 2n - 2\alpha - 2$ and $k \in \mathbb{R}^+$, the following are equivalent:

- (a) Every circle in M_α with $\langle X, X \rangle = \varepsilon_0$ and $\langle \nabla_X X, \nabla_X X \rangle = \varepsilon_1 k^2$ is a circle in \bar{M}_β .
- (b) M_α is a spherical submanifold of \bar{M}_β . \square

PROOF See [ANY90]. \blacksquare

More intuitively, the above theorem states that a spherical submanifold M is precisely a submanifold in which all circles in M are circles in the ambient space. Also note that the above theorem shows that a circle is precisely a spherical submanifold of dimension one. The following theorem is a variant of the above theorem which is known to hold (in full generality) only in the strictly pseudo-Riemannian case.

Theorem 5.4 (Circles and Spheres II [ANY90])

Let M_α be an n dimensional ($1 \leq \alpha \leq n - 1$) pseudo-Riemannian submanifold of \bar{M}_α having the same signature α . For any $\varepsilon_0 \in \{-1, 1\}$, the following are equivalent:

(a) Every geodesic in M_α with $\langle X, X \rangle = \varepsilon_0$ is a circle in \bar{M}_α .

(b) M_α is a spherical submanifold of \bar{M}_α . □

These results can be further generalized by considering more general types of curves such as helices (which we will not define here). See [Nak88] where a theorem analogous to Theorem 5.3 is proven characterizing helices in terms of geodesic submanifolds. Also in [JF94] results relating conformal circles to umbilical submanifolds are presented.

The following lemma describes how much information is required to specify a sphere. It is a partial generalization of the corresponding lemma for the Riemannian case proven in [Kas10].

Lemma 5.5 (Uniqueness of Spheres)

Suppose that M and N are connected and geodesically complete spherical submanifolds of \bar{M} both satisfying the following condition: For some $p \in M \cap N$, M and N are tangent and have the same mean curvature vectors. Then $M \equiv N$. □

PROOF Our proof is a generalization of the proof of lemma 4.14 in [O’N83, P. 105].

Let $q \in M$ be arbitrary and suppose that $\gamma(t)$ is a geodesic segment in M running from p to q . Then observe that γ is a geodesic circle in \bar{M} with velocity X_p and acceleration $\langle X, X \rangle|_p H_p^M$ at p where H^M is the mean curvature vector field of M . By the uniqueness of circles (see Lemma 5.1) and the hypothesis it follows that γ is also geodesic in N which is defined everywhere since N is geodesically complete. Note that this implies that mean curvature vector fields of M and N coincide over γ , so we denote this vector field by H .

Now suppose $Z_p \in T_p M \cap X_p^\perp$ and let Z be the parallel transport of Z_p over γ with respect to M . Since parallel transport is an isometry, $\langle Z, X \rangle = 0$. Thus by the Gauss equation,

$$\begin{aligned} \bar{\nabla}_X Z &= \nabla_X^M Z + \langle Z, X \rangle H \\ &= 0 \end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} and ∇^M is the induced Levi-Civita connection on M . Thus Z is also the parallel transport of Z_p over γ with respect to \bar{M} .

Thus the parallel transport of $T_p M \cap X_p^\perp$ to q on \bar{M} is equal to $T_q M \cap X_q^\perp$. Similarly the parallel transport of $T_p N \cap X_p^\perp$ to q on \bar{M} is equal to $T_q N \cap X_q^\perp$. Since the parallel transport on \bar{M} is uniquely determined, we deduce that $T_q M \cap X_q^\perp = T_q N \cap X_q^\perp$. Since $X_q \in T_q M, T_q N$, we conclude that $T_q M = T_q N$. Thus since M is connected, one can apply this argument to an arbitrary broken geodesic (see [O’N83]) to conclude that $M \subseteq N$.

Finally by applying the argument for M interchanged with N , we see that $M \equiv N$. ■

Let M be a space of constant curvature. We will show in this article that for every $p \in M$, non-degenerate subspace $V \subset T_p M$, and normal vector $H \in (T_p M)^\perp$ there

exists a connected and geodesically complete spherical submanifold passing through p with tangent space V and mean curvature vector H at p . In the following theorem, we will show that this property characterizes Riemannian spaces of constant curvature. For the following theorem, we say a Riemannian manifold M satisfies the *axiom of r -spheres* if: for every $p \in M$ and any r dimensional subspace $V \subset T_p M$ there exists a spherical submanifold passing through p and tangent to V .

Theorem 5.6 (Spheres in spaces of constant curvature [LN71])

Let M be a Riemannian manifold with dimension $n \geq 3$ and fix $2 \leq r < n$. Then M is a space of constant curvature iff it satisfies the axiom of r -spheres (see above). \square

PROOF See [LN71]. ■

6 Spherical Submanifolds of Spaces of Constant Curvature

In this section κ is allowed to be zero. The following optional proposition relates umbilical submanifolds to spherical ones in spaces of constant curvature.

Proposition 6.1

Any umbilical submanifold of $\mathbb{E}_\nu^n(\kappa)$ with dimension greater than one is necessarily spherical. \square

PROOF This follows from Lemma 3.2 (a) in [Che11]. ■

Here we state some properties of spherical submanifolds in spaces of constant curvature.

Proposition 6.2 (Spherical Submanifolds in Spaces of Constant Curvature)

Let $\phi : N \rightarrow \mathbb{E}_\nu^n(\kappa)^\circ$ be an isometric immersion of a pseudo-Riemannian manifold N . If N is a spherical submanifold, then

- (a) $\langle H, H \rangle$ is constant.
- (b) N is of constant curvature $\kappa + \langle H, H \rangle$

PROOF Lemma 3.2 from [Che11]. ■

7 Standard spherical submanifolds of pseudo-Euclidean space

We collect some properties of $\mathbb{E}_\nu^n(\kappa)$ in the following proposition.

Proposition 7.1

Let r denote the dilatational vector field and $r^2 = \langle r, r \rangle$. Fix $r^2 \in \mathbb{R}$, the following are true about $\mathbb{E}_\nu^n(\frac{1}{r^2})$

(a) *It is a spherical submanifold with mean curvature normal*

$$H = -\frac{r}{r^2} \quad (7.1)$$

(b) *It has constant curvature $\frac{1}{r^2}$ and is geodesically complete.* \square

PROOF The first follows from [O'N83, Lemma 4.27]. When $\dim \mathbb{E}_\nu^n(\frac{1}{r^2}) > 1$, the first result together with Proposition 6.1 shows that $\mathbb{E}_\nu^n(\frac{1}{r^2})$ is a spherical submanifold. In any case, it follows from Eq. (7.1) that $\mathbb{E}_\nu^n(\frac{1}{r^2})$ is a spherical submanifold. Hence the second result follows from Proposition 6.2 (b). It follows from lemma 4.29 in [O'N83] that $\mathbb{E}_\nu^n(\frac{1}{r^2})$ is geodesically complete. \blacksquare

We collect similar properties of \mathbb{P}_ν^n .

Proposition 7.2

The following are true about \mathbb{P}_ν^n with mean curvature vector $-a$:

(a) *It is a spherical submanifold with mean curvature normal*

$$H = -a$$

(b) *It is globally isometric to \mathbb{E}_ν^n .* \square

PROOF Consider the map ψ given by Eq. (2.3). It then follows that for $v \in TV$,

$$\psi_* v = v - \langle v, x \rangle a$$

The above equation shows that the induced metric at each point is the induced metric on V . Hence \mathbb{P}_ν^n is globally isometric to \mathbb{E}_ν^n . Now to calculate the second fundamental form, we have for $w, v \in TV$:

$$\begin{aligned} \nabla_{\psi_* w} \psi_* v &= \nabla_w v - \langle \nabla_w v, x \rangle a - \langle v, w \rangle a \\ &= \psi_* \nabla_w v - \langle v, w \rangle a \end{aligned}$$

Hence it follows that \mathbb{P}_ν^n is umbilical with mean curvature vector $-a$. Since $-a$ is covariantly constant, it follows that \mathbb{P}_ν^n is spherical. \blacksquare

8 Warped Products

In this section we define the warped product and give some properties of it which will be used in this article. The content of this section is primarily from [MRS99] where the more general notion of a twisted product was introduced. For more on warped products see [MRS99; Zeg11]. A warped product can be defined to be a special case of a twisted product as follows:

Definition 8.1 (Twisted and Warped Products)

Let $M = \prod_{i=0}^k M_i$ be a product of pseudo-Riemannian manifolds (M_i, g_i) where $\dim M_i > 0$ for $i > 0$. Suppose for $i = 0, \dots, k$, $\pi_i : M \rightarrow M_i$ is the projection map and $\rho_i : M \rightarrow \mathbb{R}^+$ is a function. The following metric g on M is called a *twisted product metric*

$$g(X, Y) = \sum_{i=0}^k \rho_i^2 g_i(\pi_{i*} X, \pi_{i*} Y) \quad \text{for } X, Y \in \mathfrak{X}(M)$$

If each ρ_i depends only on M_0 and $\rho_0 \equiv 1$ then g is called a *warped product metric* and (M, g) is called a *warped product*. The warped product is denoted by $M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_k} M_k$. M_0 is called the geodesic factor of the warped product and the M_i for $i > 0$ are called spherical factors. \square

Example 8.2 (Prototypical warped product)

The prototypical example of a warped product is the following warped product defined in (an open subset of) \mathbb{E}^n , which is the product manifold $\mathbb{R}^+ \times S^{n-1}$ equipped with the metric $g = d\rho^2 + \rho^2 \tilde{g}$ where \tilde{g} is the metric of the $(n-1)$ -sphere S^{n-1} . \square

In the above example we say the warped product $\mathbb{R}^+ \times_{\rho} S^{n-1}$ is a *warped product decomposition* of \mathbb{E}^n . In general a *warped product decomposition* of a given pseudo-Riemannian manifold M is a warped product which is (locally) isometric to M .

Each factor M_i of the product manifold induces a foliation L_i of M . For any $\bar{p} \in M$ the leaf of this foliation through \bar{p} , $L_i(\bar{p})$, is given as follows:

$$L_i(\bar{p}) := \{p \in M : p = (\bar{p}_1, \dots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \dots, \bar{p}_k), p_i \in M_i\}$$

where $\bar{p}_j = \pi_j(\bar{p})$. We let E_i denote the integrable distribution induced by L_i , then $\mathcal{E} = (E_i)_{i=1}^k$ is called the *product net* of $\prod_{i=1}^k M_i$.

We also note here that a warping function ρ_i of a warped product is only uniquely defined modulo products of constants. To elaborate, from the above definition one sees that we can multiply ρ_i^2 by any $c \in \mathbb{R}^+$ if we divide g_i by c . The geometry of the warped product is not altered by such transformations as we will see. We say that the warping functions are *normalized* (with respect to a point $\bar{p} \in M$), if for each i , $\rho_i(p) = 1$ for all $p \in L_i(\bar{p})$.

We record here some properties of the warped product:

Proposition 8.3 (Properties of the Warped Product [MRS99])

Let $\prod_{i=0}^k M_i$ be a warped product with product net $\mathcal{E} = (E_i)_{i=0}^k$.

1. \mathcal{E} is orthogonal, i.e. it satisfies: $TM = \bigoplus_{i=0}^k E_i$
2. For each $i > 0$ the distribution E_i is Killing with mean curvature normal $H_i = -\nabla(\log \rho_i)$. Hence E_0 is geodesic. \square

PROOF See proposition 2 in [MRS99]. \blacksquare

The following theorem gives a converse to the above proposition.

Theorem 8.4 (Geometric Characterization of Warped Products [MRS99])

Let $M = \prod_{i=0}^k M_i$ be a connected product manifold equipped with metric g and orthogonal product net $\mathcal{E} = (E_i)_{i=0}^k$. Then g is the metric of a warped product $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ iff E_i are Killing foliations for $i = 1, \dots, k$. \square

In particular one should note that for a warped product, the manifolds M_i are spherical submanifolds of M . This is an important observation in constructing warped products. We say a warped product is *proper* if none of the spherical factors are geodesic submanifolds. One can check (see proposition 2 in [MRS99]) that this is equivalent to requiring ρ_i to be non-constant.

The above theorem also shows that any manifold can be made into a spherical submanifold (of some other manifold). Hence this notion is only interesting with respect to a fixed manifold.

9 Warped product decompositions of Spaces of Constant Curvature

In this section we study warped product decompositions of $\mathbb{E}_\nu^n(\kappa)$ where κ may equal zero. Let $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and $\psi : M \rightarrow \mathbb{E}_\nu^n(\kappa)$ a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$. Fix $\bar{p} \in \psi(M)$. Let $H_i = -\nabla(\log \rho_i)$ be the mean curvature vector field associated to the canonical foliation L_i generated by M_i (see Proposition 8.3). Let $V_i := T_{\bar{p}_i} M_i$ for each i and $z_i := H_i|_{\bar{p}} \in V_0$ for $i > 0$. Then note that

$$T_{\bar{p}} M = \bigoplus_{i=0}^k V_i$$

It follows from corollary 2 in [MRS99] that the mean curvature vectors satisfy the following equation for $i \neq j$:

$$\langle z_i, z_j \rangle = -\kappa \tag{9.2}$$

In this case we say that ψ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ associated with the initial data $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ where $a_i := \kappa\bar{p} - z_i$.

Conversely, let $\bar{p} \in \mathbb{E}_\nu^n(\kappa)$ where $n \geq 2$ and consider the following decomposition of $T_{\bar{p}}\mathbb{E}_\nu^n(\kappa)$, $T_{\bar{p}}\mathbb{E}_\nu^n = \bigoplus_{i=0}^k V_i$ into non-trivial subspaces (hence non-degenerate) with $k \geq 1$. Suppose $z_1, \dots, z_k \in V_0$ satisfy Eq. (9.2). Let $a_i := \kappa\bar{p} - z_i$ and assume additionally that the subset of non-zero a_i are linearly independent. In this case, we say that $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ are *initial data* for a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$. We will show in this article that in a space of constant curvature there always exists a warped product decomposition associated with any given initial data. It follows from Theorem 5.6 that in the category of Riemannian manifolds with $n > 2$, this property characterizes spaces of constant curvature.

The additional condition requiring the a_i to be linearly independent trivially holds in Euclidean space and in motivating applications. The reason we make this assumption will become more apparent later. Here is an optional lemma, which is given for completeness, and hints at why we make this assumption.

Lemma 9.1

Suppose a_1, \dots, a_k are linearly independent pair-wise orthogonal lightlike vectors. Then there exist vectors b_1, \dots, b_k such that $\langle a_i, b_j \rangle = \delta_{ij}$ and $\langle b_i, b_j \rangle = 0$. \square

PROOF Suppose to the contrary that for any b_1 satisfying $\langle b_1, a_i \rangle = 0$ for $i > 1$ we have $\langle b_1, a_1 \rangle = 0$. Thus

$$\cap_{i=2}^k a_i^\perp \subseteq a_1^\perp$$

Define $T : V \rightarrow \mathbb{R}^k$ by:

$$T(v) = (\langle a_1, v \rangle, \dots, \langle a_k, v \rangle)$$

By hypothesis we have $\dim \ker T \geq n - (k - 1)$, hence $\dim \operatorname{Im} T \leq k - 1$ by the rank-nullity theorem. Thus $a_1^\perp, \dots, a_k^\perp$ are linearly dependent, a contradiction.

Thus there exists $b_1 \in \cap_{i=2}^k a_i^\perp$ with $\langle a_1, b_1 \rangle = 1$. The result then follows by induction. Indeed the next step is to find b_2 by applying the above result to $\{a_2, \dots, a_k\} \subset \operatorname{span}\{a_1, b_1\}^\perp$ making use of the fact that $\operatorname{span}\{a_1, b_1\}$ is non-degenerate by construction. \blacksquare

It has been shown by Nolker in [Nol96] that given any initial data for Riemannian spaces of constant curvature, there exists a unique warped product decomposition associated with the initial data. In this article we will show that given any initial data for a WP-decomposition of $\mathbb{E}_\nu^n(\kappa)$, there exists a WP-decomposition associated with the initial data. This WP-decomposition is probably uniquely determined but we don't use or prove this supposition.

One can also deduce that from corollary 2 in [MRS99] that the Hessian H of each warping function ρ of a space of constant curvature satisfies the following equation on the geodesic factor:

$$H(X, Y) = -\kappa\rho \langle X, Y \rangle$$

This proves the following fact:

Lemma 9.2

A space of constant non-zero curvature does not admit product decompositions. \square

10 Warped product decompositions of pseudo-Euclidean space

10.1 Spherical submanifolds of pseudo-Euclidean space

We first describe the spherical submanifolds of pseudo-Euclidean space. The following theorem is a generalization of Lemma 5 in [Nol96] to pseudo-Euclidean space.

Theorem 10.1 (Spherical submanifolds of \mathbb{E}_ν^n)

Let $\bar{p} \in$ be arbitrary, $V \subseteq \mathbb{E}_\nu^n$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \text{ind } V$ and $z \in V^\perp$. Let $\tilde{\kappa} := z^2$, $a := -z$ and $W = \mathbb{R}a \oplus V$. There is exactly one m -dimensional connected and geodesically complete spherical submanifold \tilde{N} with $\bar{p} \in \tilde{N}$, $T_{\bar{p}}\tilde{N} = V$ and having mean curvature vector at \bar{p} , z . \tilde{N} is an open submanifold of N ; N is referred to as the spherical submanifold determined by (\bar{p}, V, a) and is given as follows (where \simeq means isometric to):

(a) $a = 0$ iff N is geodesic, in this case $N \simeq \mathbb{E}_\mu^m$

$$N = \bar{p} + V$$

(b) a is timelike, then $\mu \leq \nu - 1$ and $N \simeq H_\mu^m(\tilde{\kappa})$

(c) a is spacelike, then $N \simeq S_\mu^m(\tilde{\kappa})$

For cases (b) and (c), let $c = \bar{p} - \frac{a}{\tilde{\kappa}}$ be the center of N , then N is given as follows:

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(d) a is lightlike, then $\mu \leq \nu - 1$ and $N \simeq E_\mu^m$

$$N = \bar{p} + \{p - \frac{1}{2}p^2a \mid p \in V\}$$

\square

Remark 10.2

$\tilde{N} = N$ except in the following two cases (which are anti-isometric): When $N \simeq H_0^m(\tilde{\kappa})$ or $N \simeq S_m^m(\tilde{\kappa})$, N is disconnected [O’N83, Section 4.6] and so \tilde{N} is given as follows:

$$\tilde{N} = N \cap (c + \{p \in W \mid \langle a, p \rangle > 0\}) \quad \square$$

PROOF First we note that it suffices to show that there exists a single connected and geodesically complete sphere satisfying the initial conditions. By Lemma 5.5, it must be unique.

Item (a) is clear. For Items (b) and (c), it follows from Proposition 7.1 that N is a sphere and the initial conditions are easily checked. The connectedness properties follow from lemma 4.25 in [O’N83]. It follows from lemma 4.29 in [O’N83] that N is geodesically complete.

Item (d) follows from Proposition 7.2. ■

Remark 10.3

See [Che11] for a different proof. □

Since circles are one dimensional spherical submanifolds, we can use the above theorem to describe the circles in pseudo-Euclidean space.

Example 10.4 (Proper Circles in pseudo-Euclidean space)

Suppose $(\bar{p}, \bar{V}, k\bar{Y})$ are initial conditions for a proper circle as in Lemma 5.1 with $\varepsilon_0 := \bar{V}^2 = \pm 1$, $\varepsilon_1 := \bar{Y}^2 = \pm 1$ and $\|k\bar{Y}\| \neq 0$. We now describe the circle determined by this data.

By Example 5.2 the proper circle determined by these initial conditions determine a spherical submanifold of \mathbb{E}_ν^n characterized by $(\bar{p}, \mathbb{R}\bar{V}, \varepsilon_0 k\bar{Y})$. Now let $H := \varepsilon_0 k\bar{Y}$, $\kappa := \langle H, H \rangle = \varepsilon_1 k^2$ and $c := \bar{p} + \frac{H}{\kappa} = \bar{p} - \frac{\varepsilon_0 \varepsilon_1 \bar{Y}}{k}$.

Case 1 Euclidean circle, $\gamma = \mathbb{S}^1$: $\varepsilon_0 = \varepsilon_1 = \pm 1$

$$\gamma(t) = c + \frac{1}{k}(\sin(kt)\bar{V} - \cos(kt)\bar{Y})$$

Case 2 Hyperbolic circle, $\gamma = H^1$: $\varepsilon_0 = 1, \varepsilon_1 = -1$

Case 3 de Sitter circle, $\gamma = \mathbb{S}_1^1$: $\varepsilon_0 = -1, \varepsilon_1 = 1$

In the last two cases (which are anti-isometric), γ is given as follows:

$$\gamma(t) = c + \frac{1}{k}(\sinh(kt)\bar{V} - \varepsilon_0 \varepsilon_1 \cosh(kt)\bar{Y}) \quad \square$$

One can give a similar example for geodesics and null circles.

10.2 Warped product decompositions of pseudo-Euclidean space

Our classification of the warped product decompositions of \mathbb{E}_ν^n is based on the fact that a specification of the tangent spaces and mean curvature normals of the spherical foliations of a warped product at one point \bar{p} , uniquely determines a warped product decomposition in a neighborhood of \bar{p} . We now carry out this classification as follows. Suppose $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \rightarrow \mathbb{E}_\nu^n$ is a warped product decomposition of \mathbb{E}_ν^n associated with initial data $(\bar{p}; \bigoplus_{i=0}^k V_i; -z_1, \dots, -z_k)$. By Eq. (9.2), the mean curvature vectors at \bar{p} satisfy the following equation:

$$\langle z_i, z_j \rangle = 0 \quad i \neq j$$

We now only consider the case $\nu \leq 1$ as the other signatures are straightforward generalizations of these standard ones. In this case, we will use Theorem 10.1 to classify N_i up to homothety as follows. Say $z_1, \dots, z_l = 0$ and the remaining are non-zero, then for $i = 1, \dots, l$ the N_i are pair-wise orthogonal planes passing through \bar{p} . We now consider the remaining possibilities:

- Case 1** Since the z_i are orthogonal, there is at most one lightlike direction, say z_{l+1} . The remaining lightlike z_i are proportional to z_{l+1} , but since we assume the non-zero z_i are linearly independent, we will work with only one lightlike vector z_{l+1} . Then N_{l+1} is a paraboloid isometric to Euclidean space. The orthogonality relations force the remaining z_i to be space-like and hence the remaining N_i are Euclidean spheres.
- Case 2** Similarly, at most one of the z_i can be timelike, say z_{l+1} . Then N_{l+1} is isometric to hyperbolic space. The orthogonality relations force the remaining z_i to be space-like and hence the remaining N_i are Euclidean spheres.
- Case 3** The remaining z_i are spacelike. If $\text{ind } V_0 = 1$ or $\text{ind } V_0 = 0$ in Euclidean space, then the remaining N_i are Euclidean spheres. If $\text{ind } V_0 = 0$ in Minkowski space, then $\text{ind } V_j = 1$ for precisely one $j \geq 1$, then N_j is de Sitter space while the remaining N_i are Euclidean spheres.
- Case 4** All z_i are zero. Then each N_i is an affine plane and the warped product is a product of planes.

We summarize our findings in the following theorem.

Theorem 10.5 (Warped products in \mathbb{E}^n and M^n)

Suppose $N = N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$ is a proper warped product decomposition of an open subset of \mathbb{E}_ν^n . If at most one of the N_i are intrinsically flat, then N is isometric to one of the following warped products:

If \mathbb{E}_ν^n is Euclidean space:

$$\mathbb{E}^m \times_{\rho_1} S^{n_1} \times \cdots \times_{\rho_s} S^{n_s}$$

If \mathbb{E}_ν^n is Minkowski space:

$$\begin{aligned} & M^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ & M^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ & \mathbb{E}^m \times_{\rho_1} dS^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ & M^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \end{aligned}$$

where $\nabla \rho_i, \nabla \tau_i, \nabla \lambda_i$ is a spacelike, timelike, lightlike vector field respectively. \square

The above theorem shows that there are at 1 and 4 distinct types of proper singly warped products in Euclidean and Minkowski space respectively. One can show that the multiply warped products can be built up from the singly warped products by iteratively decomposing the geodesic factor of the warped product into another warped product which is “compatible” with the original. Thus we only describe a special subset of warped products for simplicity.

The following theorem describes this interesting class of warped products. Its proof can be deduced from Theorem 7 in [Nol96]. It is a generalization of that theorem to pseudo-Euclidean space.

Theorem 10.6 (Standard Warped Products in \mathbb{E}_ν^n [Nol96])

Fix $\bar{p} \in \mathbb{E}_\nu^n$ where $n \geq 2$ and the following decomposition of $T_{\bar{p}}\mathbb{E}_\nu^n$, $T_{\bar{p}}\mathbb{E}_\nu^n = \bigoplus_{i=0}^k V_i$ into non-trivial subspaces (hence non-degenerate) with $k \geq 1$. Suppose $a_1, \dots, a_k \in V_0$ are pair-wise orthogonal. Let $\kappa_i := a_i^2$ and $\epsilon_i := \text{sgn } \kappa_i$. We consider the following warped decompositions:

non-null warped decomposition Let $\mu \geq 0$

$$\kappa_1 \leq \cdots \leq \kappa_\mu < 0 < \kappa_{\mu+1} \leq \cdots \leq \kappa_k$$

In this case, let $c = \bar{p} - \sum_{i=1}^k \frac{a_i}{\kappa_i}$ and $c_i = \bar{p} - \frac{a_i}{\kappa_i}$ for each $i = 1, \dots, k$.

null warped decomposition $k = 1$, $a_1 := a$, $\kappa_1 = a^2 = 0$ but $a \neq 0$, i.e. a is lightlike.

In this case, fix a lightlike vector $b \in V_0$ such that $\langle a, b \rangle = 1$ and let $c = \bar{p} - b$.

Now, define N_0 as follows:

$$N_0 := c + \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for all } i\}$$

Note that N_0 is an open subset of the plane determined by $(\bar{p}, V_0, 0)$. For $i = 1, \dots, k$, let N_i be the spherical submanifold of \mathbb{E}_ν^n determined by (\bar{p}, V_i, a_i) . Define

$$\rho_i : \begin{cases} N_0 & \rightarrow \mathbb{R}_+ \\ p_0 & \mapsto \langle a_i, p_0 - c \rangle = 1 + \langle a_i, p_0 - \bar{p} \rangle \end{cases}$$

For $i = 1, \dots, k$, let $W_i := \mathbb{R}a_i \oplus V_i$ and $P : \mathbb{E}_\nu^n \rightarrow W_i$ be the orthogonal projection. Then the map

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k & \rightarrow \mathbb{E}_\nu^n \\ (p_0, \dots, p_k) & \mapsto p_0 + \sum_{i=1}^k \rho_i(p_0)(p_i - \bar{p}) \end{cases} \quad (10.11)$$

is an isometry onto the following set⁸:

$$\text{Im}(\psi) := \begin{cases} c + \{p \in \mathbb{E}_\nu^n \mid \text{sgn}(P_i(p))^2 = \epsilon_i, \text{ for each } i = 1, \dots, k\} & \text{non-null case} \\ c + \{p \in \mathbb{E}_\nu^n \mid \langle a, p \rangle > 0\} & \text{null case} \end{cases}$$

$\text{Im}(\psi)$ is dense in \mathbb{E}_ν^n only for a non-null warped decomposition when each W_i for $i = 1, \dots, k$ is Euclidean or anti-isometric to a Euclidean space. \square

Remark 10.7

Note that $\rho_i(\bar{p}) = 1$ for $i = 1, \dots, k$. Also for each $p_i \in N_i$ we have $\psi(\bar{p}, \dots, p_i, \dots, \bar{p}) = p_i$, hence $\psi(\bar{p}, \dots, \bar{p}) = \bar{p}$.

If the N_i are required to be connected, then $\text{Im}(\psi)$ has to be modified slightly. For each N_i that is disconnected (see the remark following Theorem 10.1), in addition to the restriction that $\text{sgn}(P_i(p))^2 = \epsilon_i$ in the definition of $\text{Im}(\psi)$, add the restriction that $\langle a_i, P_i(p) \rangle > 0$. \square

PROOF The idea of this proof is to assume Eq. (10.11) holds and then expand it by choosing an appropriate basis for V_0 . In the expanded form we will be able to prove all the claims made in the theorem. We have the following two cases.

The non-null case: Let W_0 be the orthogonal complement of $\bigoplus_{i=1}^k \mathbb{R}a_i$ in V_0 ; which is well defined since $a_i^2 \neq 0$ for each i . Thus we have that

$$V_0 = W_0 \oplus \bigoplus_{i=1}^k \mathbb{R}a_i \quad (10.13)$$

which implies:

⁸Note that $\text{sgn } 0 = 0$, otherwise for $a \neq 0$, $\text{sgn } a$ is the sign of a .

$$\begin{aligned}
\mathbb{E}_\nu^n &= \bigoplus_{i=0}^k V_i \\
&= W_0 \oplus \bigoplus_{i=1}^k \mathbb{R}a_i \oplus \bigoplus_{i=1}^k V_i \\
&= W_0 \oplus \bigoplus_{i=1}^k (\mathbb{R}a_i \oplus V_i) \\
&= W_0 \oplus \bigoplus_{i=1}^k W_i
\end{aligned}$$

Now let $P_i : \mathbb{E}_\nu^n \rightarrow W_i$ denote the orthogonal projection for $i = 0, \dots, k$. Then from Eq. (10.13), we get the following orthogonal decomposition of V_0 which will be used extensively:

$$p = P_0 p + \sum_{i=1}^k \frac{1}{\kappa_i} \langle a_i, p \rangle a_i \quad \text{for all } p \in V_0 \quad (10.14)$$

Now we use the above decomposition of $p \in V_0$ to write $\psi(p_0, \dots, p_k)$ adapted to the following affine decomposition of \mathbb{E}_ν^n

$$\mathbb{E}_\nu^n = c + \bigoplus_{i=0}^k W_i$$

We get the following for $(p_0, \dots, p_k) \in N_0 \times \dots \times N_k$

$$\psi(p_0, \dots, p_k) = c + P_0(p_0 - c) + \sum_{i=1}^k \langle a_i, p_0 - c \rangle (p_i - c_i) \quad (10.16)$$

Now we prove that ψ is injective: Let $(p_0, \dots, p_k), (q_0, \dots, q_k) \in N_0 \times \dots \times N_k$ and suppose that $\psi(p_0, \dots, p_k) = \psi(q_0, \dots, q_k)$. From Eq. (10.16), we deduce the following:

$$\begin{aligned}
P_0(p_0 - c) &= P_0(q_0 - c) \\
\langle a_i, p_0 - c \rangle (p_i - c_i) &= \langle a_i, q_0 - c \rangle (q_i - c_i)
\end{aligned}$$

Since for each $i = 1, \dots, k$, $(p_i - c_i)^2 = (q_i - c_i)^2 = \frac{1}{\kappa_i}$ and $\langle a_i, p_0 - c \rangle, \langle a_i, q_0 - c \rangle \in \mathbb{R}^+$, we deduce that $p_i = q_i$. Then Eq. (10.14) shows $p_0 = q_0$.

Now for surjectivity: From Eq. (10.16) it's clear that $\psi(N_0 \times \dots \times N_k) \subseteq \text{Im}(\psi)$. Given $p \in \text{Im}(\psi)$, using Eq. (10.16) in conjunction with Eq. (10.14) we can readily calculate the inverse $q = \psi^{-1}(p)$ given in components as follows:

$$q_0 = c + P_0(p - c) + \sum_{i=1}^k \frac{\epsilon_i}{\sqrt{|\kappa_i|}} \|P_i(p - c)\| a_i$$

$$q_i = c_i + \frac{1}{\sqrt{|\kappa_i|}} \frac{P_i(p - c)}{\|P_i(p - c)\|} \quad i = 1, \dots, k$$

Now we show that ψ is an isometry. Note first that for $p = (p_0, \dots, p_k) \in N_0 \times \dots \times N_k$ and $v = (v_0, \dots, v_k) \in T_p(N_0 \times \dots \times N_k)$, Eq. (10.16) implies that

$$\psi_* v = P_0 v_0 + \sum_{i=1}^k \langle a_i, v_0 \rangle (p_i - c_i) + \sum_{i=1}^k \langle a_i, p_0 - c \rangle v_i$$

Hence also using the fact that:

$$\langle p_i - c_i, v_i \rangle = 0 \text{ for } i = 1, \dots, k$$

we get:

$$\begin{aligned} (\psi_* v)^2 &= (P_0 v_0)^2 + \sum_{i=1}^k (\langle a_i, v_0 \rangle (p_i - c_i))^2 + \sum_{i=1}^k (\langle a_i, p_0 - c \rangle v_i)^2 \\ &= (P_0 v_0)^2 + \sum_{i=1}^k \frac{\langle a_i, v_0 \rangle^2}{\kappa_i} + \sum_{i=1}^k \rho_i(p_0)^2 v_i^2 \\ &= (P_0 v_0 + \sum_{i=1}^k \frac{\langle a_i, v_0 \rangle}{\kappa_i} a_i)^2 + \sum_{i=1}^k \rho_i(p_0)^2 v_i^2 \\ &= v_0^2 + \sum_{i=1}^k \rho_i(p_0)^2 v_i^2 \end{aligned}$$

where the last two lines follow from the fact that $v_0 \in V_0$ and Eq. (10.14).

The null case: We have the following decomposition of V_0 :

$$V_0 = W_0 \oplus \text{span}\{a, b\}$$

where W_0 is the orthogonal complement of $\text{span}\{a, b\}$ relative to V_0 . Let P_i denote the orthogonal projection onto W_0 for $i = 0$ and onto V_1 for $i = 1$. Then for $p \in \mathbb{E}_\nu^n$:

$$p = P_0 p + \langle b, p \rangle a + \langle a, p \rangle b + P_1 p \quad (10.20)$$

and

$$p^2 = (P_0 p)^2 + 2 \langle b, p \rangle \langle a, p \rangle + (P_1 p)^2 \quad (10.21)$$

Let $c = \bar{p} - b$, $\tilde{p}_0 = p_0 - c$ and $\tilde{p}_1 = p_1 - \bar{p}$, then for $(p_0, p_1) \in N_0 \times N_1$

$$\begin{aligned} \psi(p_0, p_1) = c + P_0(\tilde{p}_0) + (\langle b, \tilde{p}_0 \rangle - \frac{1}{2} \langle a, \tilde{p}_0 \rangle (P_1(\tilde{p}_1))^2) a + \langle a, \tilde{p}_0 \rangle b \\ + \langle a, \tilde{p}_0 \rangle P_1(\tilde{p}_1) \end{aligned} \quad (10.22)$$

where the last two lines follow from Eq. (10.21).

Injectivity of ψ follows readily from Eq. (10.22).

Now for surjectivity: From Eq. (10.22) it's clear that $\psi(N_0 \times N_1) \subseteq \text{Im}(\psi)$. Given $p \in \text{Im}(\psi)$, let $\tilde{p} = p - c$, then using Eq. (10.16) in conjunction with Eq. (10.14) we can readily calculate the inverse $q = \psi^{-1}(p)$ given in components as follows:

$$\begin{aligned} q_0 = c + P_0(\tilde{p}) + (\langle b, \tilde{p} \rangle + \frac{1}{2 \langle a, \tilde{p} \rangle} (P_1(\tilde{p}))^2) a + \langle a, \tilde{p} \rangle b \\ q_1 = \bar{p} + \frac{1}{\langle a, \tilde{p} \rangle} P_1(\tilde{p}) - \frac{1}{2 \langle a, \tilde{p} \rangle^2} (P_1(\tilde{p}))^2 a \end{aligned} \quad (10.23)$$

Now we show that ψ is an isometry. Note first that for $p = (p_0, p_1) \in N_0 \times N_1$ and $v = (v_0, v_1) \in T_p(N_0 \times N_1)$, Eq. (10.22) implies that

$$\begin{aligned} \psi_* v = P_0 v_0 + (\langle b, v_0 \rangle - \frac{1}{2} \langle a, v_0 \rangle (P_1(\tilde{p}_1))^2 - \langle a, \tilde{p}_0 \rangle \langle P_1 \tilde{p}_1, P_1 v_1 \rangle) a + \langle a, v_0 \rangle b \\ + \langle a, v_0 \rangle P_1(\tilde{p}_1) + \langle a, \tilde{p}_0 \rangle P_1(v_1) \end{aligned}$$

Hence we get that:

$$\begin{aligned} (\psi_* v)^2 &= (P_0 v_0)^2 + 2 \langle b, v_0 \rangle \langle a, v_0 \rangle + \langle a, \tilde{p}_0 \rangle^2 (P_1 v_1)^2 \\ &= (\langle b, v_0 \rangle a + \langle a, v_0 \rangle b + P_0 v_0)^2 + \rho(p_0)^2 (P_1 v_1)^2 \\ &= v_0^2 + \rho(p_0)^2 v_1^2 \end{aligned}$$

where the last two lines follow from the fact that $v_0 \in V_0$, Eq. (10.21) and since $P_1 : T_{p_1} N_1 \rightarrow V_1$ is an isometry for each $p_1 \in N_1$. \blacksquare

Definition 10.8

We call ψ the warped product decomposition of \mathbb{E}_ν^n determined by $(\bar{p}; N_1, \dots, N_k)$ or by $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ as in the hypothesis of the above theorem. \square

Note that in the context of the above definition, the warped product decomposition is proper if each $a_i \neq 0$. For actual calculations we wish to work with canonical forms. The following definition will be particularly convenient.

Definition 10.9 (Canonical form for Warped products of \mathbb{E}_ν^n)

We say that a proper warped product decomposition of \mathbb{E}_ν^n determined by $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ is in canonical form if: $\bar{p} \in V_0$ and $\langle \bar{p}, a_i \rangle = 1$. \square

We note here that any proper warped product decomposition ψ of \mathbb{E}_ν^n can be brought into canonical form by the translation $\psi \rightarrow \psi - c$. This follows from the above theorem by observing that $\langle \bar{p} - c, a_i \rangle = 1$ for each $i > 0$. The following corollary gives the standard warped product decompositions of \mathbb{E}_ν^n in canonical form.

Corollary 10.10 (Canonical form for Warped products of \mathbb{E}_ν^n)

Let ψ be a proper warped product decomposition of \mathbb{E}_ν^n determined by $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ which is in canonical form.

Then the conclusions of Theorem 10.6 simplify as follows:

$$N_0 = \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for all } i\}$$

$$\rho_i = \langle a_i, p_0 \rangle$$

$$\text{Im}(\psi) = \begin{cases} \{p \in \mathbb{E}_\nu^n \mid \text{sgn}(P_i(p))^2 = \epsilon_i, \text{ for each } i = 1, \dots, k\} & \text{non-null case} \\ \{p \in \mathbb{E}_\nu^n \mid \langle a, p \rangle > 0\} & \text{null case} \end{cases}$$

For $(p_0, \dots, p_k) \in N_0 \times \dots \times N_k$, ψ has the following form:

$$\psi(p_0, \dots, p_k) = \begin{cases} P_0 p_0 + \sum_{i=1}^k \langle a_i, p_0 \rangle (p_i - c_i) & \text{non-null case} \\ P_0 p_0 + (\langle b, p_0 \rangle - \frac{1}{2} \langle a, p_0 \rangle (P_1(p_1))^2) a + \langle a, p_0 \rangle b + \langle a, p_0 \rangle P_1 p_1 & \text{null case} \end{cases} \quad \square$$

Furthermore, the following equation holds:

$$\psi(p_0, \dots, p_k)^2 = p_0^2 \quad (10.28)$$

PROOF First note that for the non-null case:

$$\begin{aligned} \langle a_i, c \rangle &= \left\langle a_i, \bar{p} - \frac{a_i}{\kappa_i} \right\rangle \\ &= 1 - \frac{\langle a_i, a_i \rangle}{\kappa_i} \\ &= 0 \end{aligned}$$

Similarly for the null-case:

$$\begin{aligned}\langle c, a \rangle &= \langle \bar{p} - b, a \rangle \\ &= 0\end{aligned}$$

Thus we see that

$$\begin{aligned}N_0 &= c + \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for all } i\} \\ &= \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for all } i\}\end{aligned}$$

The formula for $\text{Im}(\psi)$ follows similarly. Clearly $\rho_i(p_0) = \langle a_i, p_0 - c \rangle = \langle a_i, p_0 \rangle$. Now we break into cases.

The non-null case:

Note that $c \in W_0$, so $P_0 c = c$, hence

$$\begin{aligned}\psi(p_0, \dots, p_k) &= c + P_0(p_0 - c) + \sum_{i=1}^k \langle a_i, p_0 - c \rangle (p_i - c_i) \\ &= c + P_0(p_0 - c) + \sum_{i=1}^k \langle a_i, p_0 \rangle (p_i - c_i) - \sum_{i=1}^k \langle a_i, c \rangle (p_i - c_i) \\ &= P_0 p_0 + \sum_{i=1}^k \langle a_i, p_0 \rangle (p_i - c_i)\end{aligned}$$

It follows from the above equation that $\psi(p_0, \dots, p_k)^2 = p_0^2$.

The null case:

By Eq. (10.20), c can be written as follows:

$$c = P_0 c + \langle b, c \rangle a$$

Thus Eq. (10.22) reduces to

$$\begin{aligned}\psi(p_0, p_1) &= c + P_0(p_0) - P_0 c - \langle b, c \rangle a + (\langle b, p_0 \rangle - \frac{1}{2} \langle a, p_0 \rangle (P_1(\tilde{p}_1))^2) a + \langle a, p_0 \rangle b \\ &\quad + \langle a, p_0 \rangle P_1(\tilde{p}_1) \\ &= P_0(p_0) + (\langle b, p_0 \rangle - \frac{1}{2} \langle a, p_0 \rangle (P_1(p_1))^2) a + \langle a, p_0 \rangle b \\ &\quad + \langle a, p_0 \rangle P_1(p_1)\end{aligned}$$

In the last equation we used the fact that $P_1 \tilde{p}_1 = P_1 p_1$ since $\bar{p} \in V_0$.

Finally, it follows from the above equation that $\psi(p_0, p_1)^2 = p_0^2$. ■

11 Isometry groups of Spherical submanifolds of pseudo-Euclidean space*

Warped products of spaces of constant curvature are closely related to certain integrable subgroups of the isometry group due to the following fact [Zeg11]:

Proposition 11.1 (Lifting isometries from Killing distributions)

Let $M = B \times_\rho F$ be a warped product and suppose $\tilde{f} : F \rightarrow F$ is an isometry of F . Then the lift f defined by

$$f(x, y) := (x, \tilde{f}(y)), \quad (x, y) \in B \times F$$

is an isometry of M . □

Theorem 5.1 in [Zeg11] shows conversely that given a certain integrable group action on a pseudo-Riemannian manifold M , one can obtain a warped product whose spherical foliation is invariant under the action of the group. Hence in spaces of constant curvature one can show that the above property characterizes warped products. In view of this, in this section we state the isometry groups which preserve the spherical submanifolds of pseudo-Euclidean space.

The isometry groups of H_ν^n and S_ν^n are well documented, see for example [O’N83, section 9.2]. In this section we will describe the isometry group of \mathbb{P}_ν^n . This is given in [Nol96, lemma 6] for the case when $\nu = 0$; that proof should generalize easily. Although, we will give a different proof (motivated by Nolker’s results) using our knowledge of warped product decompositions and Proposition 11.1.

We denote the homogeneous isometry group (i.e. orthogonal group) of $\mathbb{E}_{\nu+1}^{n+2}$ by $O_{\nu+1}(n+2)$ (see [O’N83]). Then we have the following:

Proposition 11.2

Let $-a$ be the mean curvature vector of \mathbb{P}_ν^n . The isometry group of \mathbb{P}_ν^n is:

$$I(\mathbb{P}_\nu^n) = \{T \in O_{\nu+1}(n+2) \mid Ta = a\}$$

Furthermore suppose we fix an embedding of \mathbb{E}_ν^n by fixing a subspace $V \simeq \mathbb{E}_\nu^n$, then for $p \in V$ and $\tilde{p} \in V^\perp$ we have the following Lie group isomorphism:

$$\phi : \begin{cases} O(V) \ltimes V & \rightarrow I(\mathbb{P}_\nu^n) \\ (B, v) & \mapsto \phi(B, v) \end{cases}$$

where

$$\phi(B, v)(p + \tilde{p}) = \tilde{p} + Bp + \langle a, \tilde{p} \rangle v - (\langle Bp, v \rangle + \frac{1}{2} \langle a, \tilde{p} \rangle v^2) a$$

□

PROOF Consider the warped product decomposition:

$$\psi(p_0, p) = \langle a, p_0 \rangle b + \langle a, p_0 \rangle p + (\langle b, p_0 \rangle - \frac{1}{2} \langle a, p_0 \rangle p^2) a$$

for $p_0 \in N_0$ and $p \in V$. Note that

$$\psi(b, p) = b + p - \frac{1}{2} p^2 a$$

is a map onto \mathbb{P}_ν^n . As in Eq. (10.23), one can deduce that the inverse of ψ is

$$\begin{aligned} q_0 &= (\langle b, p \rangle + \frac{1}{2 \langle a, p \rangle} (Pp)^2) a + \langle a, p \rangle b \\ q_1 &= \frac{1}{\langle a, p \rangle} Pp \end{aligned}$$

Let $B \in O(V)$, $v \in V$ and define $Tp = Bp + v$ for $p \in V$. Now define \hat{T} by:

$$\hat{T} : \begin{cases} \mathbb{E}_{\nu+1}^{n+2} & \rightarrow \mathbb{E}_{\nu+1}^{n+2} \\ p & \mapsto \psi(p_0, Tp_1) \end{cases}$$

Since ψ is a warped product decomposition, it follows by Proposition 11.1 that \hat{T} induces an isometry of some open subset of $\mathbb{E}_{\nu+1}^{n+2}$ onto itself. We will now calculate \hat{T} explicitly.

For arbitrary $x \in \mathbb{E}_{\nu+1}^{n+2}$ write $x = p + \tilde{p}$ where $p \in V$ and $\tilde{p} \in V^\perp$.

$$\begin{aligned} (Tq_1)^2 &= \left\| \frac{1}{\langle a, x \rangle} Bp + v \right\|^2 \\ &= \left(\frac{1}{\langle a, x \rangle^2} (Px)^2 + \frac{2}{\langle a, x \rangle} \langle Bp, v \rangle + v^2 \right) \end{aligned}$$

$$\begin{aligned} \psi(q_0, Tq_1) &= \langle a, q_0 \rangle b + \langle a, q_0 \rangle Tq_1 + (\langle b, q_0 \rangle - \frac{1}{2} \langle a, q_0 \rangle (Tq_1)^2) a \\ &= \langle a, x \rangle b + \langle a, x \rangle Tq_1 + (\langle b, x \rangle + \frac{1}{2 \langle a, x \rangle} (Px)^2 - \frac{1}{2} \langle a, x \rangle (Tq_1)^2) a \\ &= \langle a, x \rangle b + \langle a, x \rangle Tq_1 + (\langle b, x \rangle - (\langle Bp, v \rangle + \frac{1}{2} \langle a, x \rangle v^2)) a \\ &= \langle b, x \rangle a + \langle a, x \rangle b + Bp + \langle a, x \rangle v - (\langle Bp, v \rangle + \frac{1}{2} \langle a, x \rangle v^2) a \\ &= \tilde{p} + Bp + \langle a, x \rangle v - (\langle Bp, v \rangle + \frac{1}{2} \langle a, x \rangle v^2) a \end{aligned}$$

Hence if $p := Px$ and $\tilde{p} := (I - P)x$ then

$$\hat{T}x = \tilde{p} + Bp + \langle a, x \rangle v - (\langle Bp, v \rangle + \frac{1}{2} \langle a, x \rangle v^2) a$$

Thus since \hat{T} is a linear isometry of $\mathbb{E}_{\nu+1}^{n+2}$ it follows that $\hat{T} \in O(\mathbb{E}_{\nu+1}^{n+2})$. Also \hat{T} clearly fixes a so $\hat{T} \in I(\mathbb{P}_\nu^n)$.

Let the map ϕ be as in the hypothesis. Note that $\phi(B, v) = \hat{T}$. ϕ is a Lie group homomorphism, since

$$\begin{aligned} (\hat{T}S)x &= \psi(x_0, TSx_1) \\ &= \psi((\hat{S}x)_0, T((\hat{S}x)_1)) \\ &= \hat{T}\hat{S}x \end{aligned}$$

By definition of \hat{T} it follows that ϕ is injective.

To show that ϕ is surjective, fix $T \in I(\mathbb{P}_\nu^n)$. Consider the decomposition:

$$p = \langle a, p \rangle b + \langle b, p \rangle a + Pp, \quad p \in \mathbb{E}_\nu^n$$

Using the fact that $(Tp)^2 = p^2$ with the above decomposition we obtain the following equations:

$$\begin{aligned} p^2 &= (Tp)^2 = 2 \langle a, Tp \rangle \langle b, Tp \rangle + (PTp)^2 \\ p \in V &\Rightarrow p^2 = (PTp)^2 \\ p = b &\Rightarrow 0 = b^2 = 2 \langle b, Tb \rangle + (PTb)^2 \\ p = \psi(b, \tilde{p}) &\Rightarrow 0 = p^2 = 2 \langle b, T\tilde{p} \rangle + \langle PTb, PT\tilde{p} \rangle \end{aligned}$$

The second equation implies that $PT \in O(V)$. We claim that $\phi(PT, PTb) = T$. This can be seen by decomposing the action of T with respect to the above decomposition and then using the last three equations and the fact that $T \in I(\mathbb{P}_\nu^n)$.

Hence ϕ is a Lie group isomorphism. ■

We also note that if $\psi : \mathbb{E}_\nu^n \rightarrow \mathbb{P}_\nu^n$ is the standard embedding from Eq. (2.3), then ψ is equivariant, i.e. in the notation of the proof $\psi \circ T(p) = \hat{T} \circ \psi(p)$.

12 Warped Product decompositions of Spherical submanifolds of Pseudo-Euclidean space

12.1 Spherical submanifolds of $\mathbb{E}_\nu^n(\kappa)$

In this section we will classify the spherical submanifolds of $\mathbb{E}_\nu^n(\kappa)$. In particular we will show that they all have the form $\mathbb{E}_\nu^n(\kappa) \cap (\bar{p} + W)$ for some $\bar{p} \in \mathbb{E}_\nu^n$ and some subspace W . Although not all spherical submanifolds will have this form since we are only considering the case of pseudo-Riemannian manifolds. We will see that all spherical submanifolds of $\mathbb{E}_\nu^n(\kappa)$ arise as restrictions of spherical submanifolds of \mathbb{E}_ν^n .

The following lemma concerns a submanifold N of $\mathbb{E}_\nu^n(\kappa)$. We denote by H' the mean curvature normal of N in $\mathbb{E}_\nu^n(\kappa)$ and H the mean curvature normal of N in \mathbb{E}_ν^n . Similar definitions hold for the second fundamental forms h' and h . As usual r denotes the dilatational vector field.

Lemma 12.1

If N is a submanifold of $\mathbb{E}_\nu^n(\kappa)$ then the following equations hold:

$$h(X, Y) = h'(X, Y) - \langle X, Y \rangle \frac{r}{r^2}$$

$$H = H' - \frac{r}{r^2} \tag{12.2}$$

In particular, N is an umbilical submanifold of $\mathbb{E}_\nu^n(\kappa)$ iff it is an umbilical submanifold of \mathbb{E}_ν^n . In fact, N is a spherical submanifold of $\mathbb{E}_\nu^n(\kappa)$ iff it is a spherical submanifold of \mathbb{E}_ν^n . \square

PROOF These formulas follow from lemma 3.5 and corollary 3.1 in [Che11]. \blacksquare

Now we consider the problem of finding the sphere in $\mathbb{E}_\nu^n(\kappa)$ passing through a point \bar{p} with tangent space V and mean curvature normal z at \bar{p} . We make this precise as follows.

Let $\bar{p} \in \mathbb{E}_\nu^n(\kappa)$ be arbitrary, $V \subset T_{\bar{p}}\mathbb{E}_\nu^n(\kappa)$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \text{ind } V$ and $z \in V^\perp \cap T_{\bar{p}}\mathbb{E}_\nu^n(\kappa)$.

Now let $a := \kappa\bar{p} - z$. Then assuming this data defines a submanifold of $\mathbb{E}_\nu^n(\kappa)$, we use Eq. (12.2) to obtain the mean curvature normal in \mathbb{E}_ν^n at \bar{p} , which is given as follows:

$$z - \kappa\bar{p} = -a$$

Then this determines a sphere in \mathbb{E}_ν^n with initial data (\bar{p}, V, a) by Theorem 10.1. Note that $a \neq 0$. In the following theorem we will show that this sphere in \mathbb{E}_ν^n is in fact the sphere in $\mathbb{E}_\nu^n(\kappa)$ determined by (\bar{p}, V, a) . First let $W := \mathbb{R}a \oplus V$ and $\tilde{\kappa} := a^2$.

Theorem 12.2 (Spherical submanifolds of $\mathbb{E}_\nu^n(\kappa)$)

There is exactly one m -dimensional connected and geodesically complete spherical submanifold \tilde{N} of $\mathbb{E}_\nu^n(\kappa)$ with $\bar{p} \in \tilde{N}$, $T_{\bar{p}}\tilde{N} = V$ and having mean curvature vector at \bar{p} , z . \tilde{N} is an open submanifold of N ; $N = \mathbb{E}_\nu^n(\kappa) \cap (\bar{p} + W)$ is the spherical submanifold determined by (\bar{p}, V, a) in $\mathbb{E}_\nu^n(\kappa)$ and \mathbb{E}_ν^n . In fact, N can be given explicitly as follows (where \simeq means isometric to):

- (a) a is timelike, then $\mu \leq \nu - 1$ and $N \simeq H_\mu^m(\tilde{\kappa})$
- (b) a is spacelike, then $N \simeq S_\mu^m(\tilde{\kappa})$

For cases (b) and (c), let $c = \bar{p} - \frac{a}{\tilde{\kappa}}$ be the center of N , then N is given as follows:

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(c) a is lightlike, then $\mu \leq \nu - 1$ and $N \simeq \mathbb{E}_\mu^m$

$$N = \bar{p} + \{p - \frac{1}{2}p^2a \mid p \in V\} \quad \square$$

Remark 12.3

The relationship between \tilde{N} and N follows from Remark 10.2, since the above spheres are the spheres in \mathbb{E}_ν^n determined by (\bar{p}, V, a) in Theorem 10.1. \square

Remark 12.4

N is a geodesic submanifold of $\mathbb{E}_\nu^n(\kappa)$ iff $z = 0$ iff W intersects the origin. \square

PROOF First we note that it suffices to show that there exists a single connected and geodesically complete sphere satisfying the initial conditions. By Lemma 5.5, it must be unique.

The three definitions of N given above follow directly from Theorem 10.1 with initial data (\bar{p}, V, a) . Hence the relevant intrinsic properties of N follow from Theorem 10.1. For the remainder of the proof we will assume N is given by those definitions, and we will prove the following.

Claim 12.4.1

$N = \mathbb{E}_\nu^n(\kappa) \cap (\bar{p} + W)$ \square

PROOF Note that the following equations are satisfied: $\langle \bar{p}, \bar{p} \rangle = \frac{1}{\kappa}$, $\langle a, \bar{p} \rangle = 1$

First we consider the case of Items (a) and (b). We can always write $p = c + \tilde{p}$ where $\tilde{p} \in W$. Also note that the following holds:

$$\begin{aligned} \langle c, c \rangle &= \langle \bar{p}, \bar{p} \rangle - 2 \left\langle \bar{p}, \frac{a}{\tilde{\kappa}} \right\rangle + \frac{1}{\tilde{\kappa}^2} \langle a, a \rangle \\ &= \frac{1}{\kappa} - 2 \frac{1}{\tilde{\kappa}} + \frac{1}{\tilde{\kappa}} \\ &= \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}} \end{aligned}$$

Then since $\langle c, a \rangle = 0$, we have

$$\begin{aligned} \langle p, p \rangle &= c^2 - 2 \langle c, \tilde{p} \rangle + \tilde{p}^2 \\ &= \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}} + \tilde{p}^2 \end{aligned}$$

The above equation shows that $p \in \mathbb{E}_\nu^n(\kappa)$ iff $\tilde{p} \in W(\tilde{\kappa})$, which proves the result.

Now for Item (c). We can always write $p = \bar{p} + v + wa$ where $v \in V$ and $w \in \mathbb{R}$. Hence

$$\langle p, p \rangle = \frac{1}{\kappa} + v^2 + 2w$$

The above equation shows that $p \in \mathbb{E}_\nu^n(\kappa)$ iff $w = -\frac{1}{2}v^2$, which proves the result. \blacksquare

Thus we have shown that N is a spherical submanifold of \mathbb{E}_ν^n contained in $\mathbb{E}_\nu^n(\kappa)$. It then follows from Lemma 12.1 that N is a spherical submanifold of $\mathbb{E}_\nu^n(\kappa)$ with mean curvature normal z at \bar{p} . Furthermore by Proposition 6.2 (b), this sphere is of constant curvature $\kappa + z^2 = a^2 = \tilde{\kappa}$. ■

Now we mention when we can restrict a sphere in \mathbb{E}_ν^n to one in $\mathbb{E}_\nu^n(\kappa)$. Suppose $(\bar{p}, V, -z)$ determines a sphere in \mathbb{E}_ν^n with $\bar{p} \in \mathbb{E}_\nu^n(\kappa)$ and $V \subset T_{\bar{p}}\mathbb{E}_\nu^n(\kappa)$. Then define z' as

$$z' := z + \kappa \bar{p} \in V^\perp$$

We know that $\bar{p} \in V^\perp$ and $z \in V^\perp$ by hypothesis. In order for $\langle z', \bar{p} \rangle = 0$, we must additionally assume $\langle z, \bar{p} \rangle = -1$. In this case, $(\bar{p}, V, -z)$ define initial data for a sphere in $\mathbb{E}_\nu^n(\kappa)$. It follows from the above theorem that this sphere is simultaneously the sphere in \mathbb{E}_ν^n and in $\mathbb{E}_\nu^n(\kappa)$ determined by $(\bar{p}, V, -z)$.

12.2 Warped Product decompositions of Spherical submanifolds of Pseudo-Euclidean space

Suppose $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \rightarrow \mathbb{E}_\nu^n(\kappa)$ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ associated with initial data $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ where each $a_i = \kappa \bar{p} - z_i$. By Eq. (9.2), the mean curvature vectors at \bar{p} satisfy the following equation:

$$\langle z_i, z_j \rangle = -\kappa \quad i \neq j$$

By Theorem 12.2, $L_i(\bar{p})$ is a spherical submanifold of \mathbb{E}_ν^n determined by (\bar{p}, V_i, a_i) . Note that $a_i \neq 0$. Furthermore the above equation implies that

$$\langle a_i, a_j \rangle = 0 \quad i \neq j$$

Also recall that by assumption, the a_i are linearly independent. Thus the initial data $(\bar{p}; (\mathbb{R}\bar{p} \oplus V_0) \oplus V_1 \oplus \cdots \oplus V_k; a_1, \dots, a_k)$ determines a proper warped product decomposition of the ambient space \mathbb{E}_ν^n . Furthermore, we note that this warped product decomposition is in canonical form; the canonical form was specifically designed to have this property. We now consider the converse problem of restricting a warped product decomposition in \mathbb{E}_ν^n to $\mathbb{E}_\nu^n(\kappa)$. The following theorem shows that this is always possible when the warped product in \mathbb{E}_ν^n is proper and in canonical form:

Theorem 12.5 (Restricting Warped products to $\mathbb{E}_\nu^n(\kappa)$)

Let ψ be a proper warped product decomposition of \mathbb{E}_ν^n associated with $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ which is in canonical form. Suppose $\kappa^{-1} := \bar{p}^2 \neq 0$ and let $N' := N_0(\kappa) \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$. Note that $N_0(\kappa)$ is an open subset of the sphere in $\mathbb{E}_\nu^n(\kappa)$ determined by

$(\bar{p}, (\bar{p}^\perp \cap V_0), 0)$. Then $\phi : N' \rightarrow \mathbb{E}_\nu^n(\kappa)$ defined by $\phi := \psi|_{N'}$ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ determined by $(\bar{p}; (\bar{p}^\perp \cap V_0) \bigoplus_{i=1}^k V_i; a_1, \dots, a_k)$.

Furthermore for any point $p \in \text{Im}(\psi)$ with $p^2 \neq 0$, the leaf of the foliation induced by N_i , $L_i(p)$, is simultaneously a sphere in \mathbb{E}_ν^n and $\mathbb{E}_\nu^n(\frac{1}{p^2})$. Also ψ is in canonical form at every $p \in \text{Im}(\psi)$. \square

PROOF By Eq. (10.28) in Corollary 10.10 it follows that ϕ is a diffeomorphism onto $\phi(N') \subseteq \mathbb{E}_\nu^n(\kappa)$. Clearly the restriction of the metric on N to N' is still a warped product metric. Hence it follows that ϕ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$, i.e. an isometry from a warped product. Furthermore by Theorem 12.2 it follows that for each $i > 0$, N_i is also the sphere in $\mathbb{E}_\nu^n(\kappa)$ determined by (\bar{p}, V_i, z_i) .

Now for the last point, fix $p \in \text{Im}(\psi)$ with $p^2 \neq 0$. Let \tilde{r} be the dilatational vector field in N_0 and $r := \psi_* \tilde{r}$. Can show that r is also the dilatational vector field in \mathbb{E}_ν^n (e.g. see Eq. (10.11)). Now if $\rho_i = \langle \tilde{r}, a_i \rangle$, then it can be shown using results in [MRS99] that the mean curvature vector H_i is:

$$H_i = -\frac{a_i}{\rho_i}$$

Hence $\langle \tilde{r}, -H_i \rangle = 1$. Thus at p , by making the identification $r = p$, we see that TN_i is orthogonal to $p = \psi_* \tilde{p}$ and $\langle \tilde{p}, -H_i \rangle = 1$. It follows from the discussion following Theorem 12.2 that $L_i(p)$ is also a sphere in $\mathbb{E}_\nu^n(\frac{1}{p^2})$. \blacksquare

Remark 12.6 (Connectedness)

Remark 10.7 gives the appropriate modifications of $\text{Im}(\psi)$ when each N_i for $i > 0$ are required to be connected. When $N_0(\kappa) \simeq H_0^m(\tilde{\kappa})$ or $N_0(\kappa) \simeq S_m^m(\tilde{\kappa})$, $N_0(\kappa)$ is disconnected [O'N83, Section 4.6] and so we modify $N_0(\kappa)$ as follows: By Theorem 12.2 it follows that $N_0(\kappa)$ is an open subset of the sphere in \mathbb{E}_ν^n determined by $(\bar{p}, (\bar{p}^\perp \cap V_0), \kappa \bar{p})$. Thus to enforce connectedness, it follows by Remark 10.2 that we must replace $N_0(\kappa)$ with

$$N_0(\kappa) \cap \{p \in V_0 \mid \langle \kappa \bar{p}, p \rangle > 0\}$$

PROOF $a = \kappa \bar{p}$, $\tilde{\kappa} = a^2 = \kappa$

$$\begin{aligned} c &= \bar{p} - \frac{a}{\tilde{\kappa}} \\ &= 0 \end{aligned} \quad \blacksquare$$

Now we show the effect of this on $\phi(N')$ when $\nu = 1$.

Case 1 a_i is time-like for some i

$N_0(\kappa)$ is automatically connected since $N_0(\kappa) \subset \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for each } i\}$, then $N_0(\kappa) \subset \{p \in V_0 \mid \langle \kappa \bar{p}, p \rangle > 0\}$ since $\langle a_i, \kappa \bar{p} \rangle = \kappa < 0$ (see [O'N83, P. 143] and Nolker's proof of the hyperbolic case).

Case 2 null case, $a := a_1$ is light-like

$N_0(\kappa)$ is connected here as well. First observe that it follows from the equation for ψ in Corollary 10.10 that

$$\langle a, \psi(p_0, p_1) \rangle = \langle a, p_0 \rangle > 0$$

Thus it follows that $N_0(\kappa)$ and $\phi(N')$ are in the time cone opposite to a (see remarks preceding Nolker's proof of the hyperbolic case). Thus it follows that $N_0(\kappa) \subset \{p \in V_0 \mid \langle \kappa \bar{p}, p \rangle > 0\}$, so $N_0(\kappa)$ and hence $\phi(N')$ are connected.

In this case $\phi(N')$ is the maximal connected component of $\mathbb{E}_\nu^n(\kappa)$ passing through \bar{p} .

Case 3 a_i is space-like for each i

First observe that it follows from the proof of Corollary 10.10 that $c = P_0 c \in W_0$ and $p_i - c_i \in W_i$ for $i > 0$, hence

$$\begin{aligned} \langle c, \psi(p_0, \dots, p_k) \rangle &= \left\langle c, P_0 p_0 + \sum_{i=1}^k \langle a_i, p_0 \rangle (p_i - c_i) \right\rangle \\ &= \langle c, P_0 p_0 \rangle \\ &= \langle c, p_0 \rangle \end{aligned}$$

Also since $\langle c, a_i \rangle = 0$, we have that

$$\begin{aligned} \langle c, c \rangle &= \langle c, \bar{p} \rangle \\ &= \left\langle \bar{p} - \sum_{i=1}^k \frac{a_i}{\kappa_i}, \bar{p} \right\rangle \\ &= \frac{1}{\kappa} - \sum_{i=1}^k \frac{1}{\kappa_i} \\ &< 0 \end{aligned}$$

In other words, c is time-like. Also the above equation shows that $\langle c, \kappa \bar{p} \rangle > 0$, thus c and $\kappa \bar{p}$ are in opposite time cones (see [O'N83, P. 143]). Hence,

$$\{p \in V_0 \mid \langle \kappa \bar{p}, p \rangle > 0\} = \{p \in V_0 \mid \langle \kappa c, p \rangle > 0\}$$

Thus since $\langle c, \psi(p_0, \dots, p_k) \rangle = \langle c, p_0 \rangle$, we see that $\phi(N')$ becomes

$$\phi(N') \cap \{p \in \mathbb{E}_\nu^n \mid \langle \kappa c, p \rangle > 0\}$$

□

In the following corollary we show how to obtain any warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ by restricting an appropriate warped product decomposition of \mathbb{E}_ν^n . The “appropriate” warped product decomposition of \mathbb{E}_ν^n to restrict follows from the discussion preceding the above theorem. Thus together with the above theorem, we have the following corollary:

Corollary 12.7 (Warped product decompositions of $\mathbb{E}_\nu^n(\kappa)$)

Suppose $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$ define initial data for a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$.

Let ϕ be the warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ given in the above theorem by restricting the warped product decomposition of \mathbb{E}_ν^n with initial data $(\bar{p}; (\mathbb{R}\bar{p} \oplus V_0) \bigoplus_{i=1}^k V_i; a_1, \dots, a_k)$.

Then ϕ is a warped product decomposition of $\mathbb{E}_\nu^n(\kappa)$ determined by $(\bar{p}; \bigoplus_{i=0}^k V_i; a_1, \dots, a_k)$. \square

We now mention which warped product decompositions are possible in $\mathbb{E}_\nu^n(\kappa)$. We do this by finding out when it’s possible to restrict a warped product on the ambient space. Given a warped product $(V_0 \oplus V_1 \oplus \dots \oplus V_k; a_1, \dots, a_k)$ passing through an arbitrary point in \mathbb{E}_ν^n , in order to restrict it to $\mathbb{E}_\nu^n(\kappa)$, we need it to pass through a point $\bar{p} \in V_0$ with $\bar{p}^2 = \kappa$ satisfying $\langle \bar{p}, a_i \rangle = 1$. So for a fixed $\kappa \neq 0$, we enumerate the distinct warped products in \mathbb{E}_ν^n , expand $\bar{p} \in V_0$ so that $\langle \bar{p}, a_i \rangle = 1$ and determine if it’s possible for $\bar{p}^2 = \kappa$. By making use of Theorem 10.5, we have the following results:

Theorem 12.8 (Warped products in Spherical submanifolds of \mathbb{E}^n and M^n)

Suppose $N = N_0 \times_{\rho_1} N_1 \times \dots \times_{\rho_k} N_k$ is a warped product decomposition of an open subset of a spherical submanifold of \mathbb{E}^n or M^n . This warped product is necessarily proper. If at most one of the N_i are intrinsically flat, then N is isometric to one of the following warped products:

In S^n :

$$S^m \times_{\rho_1} S^{n_1} \times \dots \times_{\rho_s} S^{n_s}$$

In dS^n :

$$\begin{aligned} & dS^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \\ & dS^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \\ & S^m \times_{\rho_1} dS^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \\ & dS^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \end{aligned}$$

In H^n :

$$\begin{aligned} & H^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \\ & H^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \\ & H^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \dots \times_{\rho_s} S^{n_s} \end{aligned}$$

where $\nabla \rho_i, \nabla \tau_i, \nabla \lambda_i$ is a spacelike, timelike, lightlike vector field respectively. \square

PROOF For the proof that the warped products are proper, see Lemma 9.2. \blacksquare

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